Midterm #1

10/2/2013

Instructions: Refer to the provided probability tables as necessary. You may use a calculator, and one sheet of notes. You will never be penalized for showing work, but if what is asked for can be computed directly, points awarded will depend primarily on the correctness of your numerical answer. The exam is out of 80 points, so you get 20 points for free. Good luck!

Problem 1 (10 points) There is a horrible disease spreading. If you get the disease, it will turn you into a shrubbery. There are two different tests for the disease – one tests your blood and the other tests your saliva. The tests are very rarely in error. You see on the news that the probability that a randomly selected person will test positive for the first test is 0.2 and the probability that a randomly selected person will test positive for the second test is also 0.2.

Your friend intends to have both tests done. Knowing that you are an expert in probability, he asks you: "What is the probability that both tests will turn out positive?"

a. What would the correct answer be if the test results were independent?

$.2 * .2 = .04$

b. Do you think that the two test results are actually independent? Explain.

No, it would seem to make sense that if one test shows a positive result, the other is more likely to as well, since both are testing for the same disease.

c. Do you think that the right answer to your friend's question is higher or lower than what you guessed in part a.? Explain in the context of part b.

If the tests are not independent, then the probability both turn out positive will be higher than .04. To see this, imagine that not only are they not independent, but that they are perfectly correlated, meaning they both always return the same result. In this case, the probability both are positive is .2,

Problem 2 (15 points) Find the following probabilities.

a. X is a normal random variable, with mean 10 and standard deviation .5. Find $P(10 \leq X \leq 11)$.

.4772

b. X is a uniform random variable on the interval $[50, 100]$. Find $P(40 \leq X \leq 70)$.

.4

c. X is a Poisson random variable, with expected value $\mu = 5$. Find $P(X \geq 3)$.

.8753

Problem 3 (10 points) To pass a test, you must correctly answer at least 10 of the 20 questions. You have studied enough so that you can correctly answer any one question with probability p. Assume that questions are drawn independently from a common pool, so that the probability of a correct answer is the same for every question, but answering one question correctly makes it no more or less likely that you answer another correctly.

a. Find the probability that you pass the exam if $p = .5$.

.5881

b. Find the probability that you pass the exam if $p = .6$.

.8725

c. Find the probability that you pass the exam if $p = .7$.

.9829

d. Based on your answers to a-c, does it appear that the probability of passing is increasing at a constant, increasing or decreasing rate in p? What is the intuition?

It appears that the probability of passing is going up at a decreasing rate, indicating decreasing returns to studying more. The idea is that a student right at the margin can realize a large gain from a small increase in p, but as p increases, the expected outcome is farther and farther over the threshold for passing, so there is a smaller and smaller corresponding increase in probability of passing. Another way to think about it is that if a student's goal is only to pass, there is little reward to studying so much that correctly answering almost all of questions is likely.

Problem 4 (15 points) You run a mutual fund that holds shares in many firms, but sometimes the firms go bankrupt. Assume that the number of bankruptcies follows a Poisson process with a mean of 2.5 bankruptcies each year.

a. What is the probability of no bankruptcies this year?

.0821

b. What is the probability of no bankruptcies for three consecutive years?

$.0821^3 = .00055$

c. Suppose the time between successive bankruptcies follows an exponential distribution, with an average time of .4 years. Let X measure the time between bankruptcies (in years).

i. Suppose a company has just gone bankrupt. What is the probability that it will be more than one year before the next bankruptcy?

.0821

ii. What is the probability that it will be more than four years before the next bankruptcy?

.0000454

Problem 5 (10 points) The weight of luggage carried onto a plane by passengers is normally distributed with a mean of 20 KG and a standard deviation of 6 KG.

a. What is the probability that a passenger's luggage weighs 25 KG?

.2023

b. Passengers are "fast-tracked" if they have luggage weighing less than 10 KG. What percentage of passengers are fast tracked?

.0478

c. The airline wants to set the maximum weight limit so that only 2.5% of passengers have to pay an overweight luggage fee. What limit should it set?

31.7598 KG

d. If the plane has 100 passengers, what is the probability that the average weight of their luggage is less than 23 KG?

The probability is very close to 0 (the exact answer is .00000028665, but any answer that notes that the probability is very small will receive full points).

Problem 6 (10 points) A new test for Bennin's Disease, which afflicts 1 in 10,000 people, has been developed. If administered to a patient with Bennin's Disease, the probability of a positive test is 99.9%, while if administered to someone who doesn't have the disease, the probability of a (false) positive is 1%. Given a positive test result, what is the probability that patient has Bennin's disease?

Applying Bayes' rule,

$$P(\text{has disease}|\text{positive test}) =$$
$$\frac{P(\text{has disease}) * P(\text{tests positive}|\text{has disease})}{P(\text{has disease}) * P(\text{tests positive}|\text{has disease}) + P(\text{does not have disease}) * P(\text{tests positive}|\text{does not have disease})}$$
$$= \frac{\frac{1}{10000} * .999}{\frac{1}{10000} * .999 + \frac{9999}{10000} * .01}$$
$$= .009892$$

So there is less than a 1% chance that a patient who tests positive has Bennin's disease.

Problem 7 (10 points) The tables below list the possible outcomes for a random variable X, but do not give their probabilities. Fill in the probabilities as directed. You may assume that X does not take values outside of the set $\{1, 2, 3, 4, 5, 6\}$.

a. Fill in the probabilities in the second column so that the expected value of X is equal to 4.5. There are many possible correct answers. Show your work.

value	probability
1	
2	
3	
4	
5	
6	

There are many correct answers. One simple correct answer is $P(4) = \frac{1}{2}$, $P(5) = \frac{1}{2}$, and $P(1) = P(2) = P(3) = P(6) = 0$. Then, $E(X) = \frac{1}{2} * 4 + \frac{1}{2} * 5 = 4.5$.

b. Fill in the probabilities in the second column so that the expected value of X is equal to 2 and the standard deviation of X is equal to 0. Show your work.

value	probability
1	
2	
3	
4	
5	
6	

This question has only one correct answer, namely $P(2) = 1$ and $P(1) = P(3) = P(4) = P(5) = P(6) = 0$.

c. Fill in the probabilities in the second column so that the expected value of X is equal to 4 and the standard deviation of X is equal to 1. Show your work.

value	probability
1	
2	
3	
4	
5	
6	

Bonus Problem (+5 points) Every day you pass by two police officers checking for speeders. The first police officer catches speeders 1% of the time. The second police officer catches speeders 2% of the time. The two officers act independently of each other. You travel on the highway for 25 days and you speed every day. What is the probability of getting at least one ticket?

First, we must determine the probability of being stopped on any one given day. There is a 99% chance of getting by the first policeman unmolested, and a 98% chance of getting by the second. Therefore, the probability of getting a ticket on any given day is $1 - .99 * .98 = .0298$. Now, what is the probability of getting at least one ticket in 25 days? First, the probability of getting zero tickets for 25 consecutive days is $P(0 \text{ tickets}) = (1 - .0298)^2 5 = .4694$, and so the probability of getting at least one ticket is one minus this, or .5306.