Economics 590, fall 2007

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Homework 6

due 11/8/2007

Problem 1 (Monotonic transformations). *(Jehle and Reny, page 510) Let $F(z)$ be an increasing function of the single variable z. For the composite function, $F(f(x))$. Show that x^* is a local maximum (minimum) of $f(x)$ if and only if x^* is a local maximum (minimum) of $F(f(x))$.*

Problem 2 (Unconstrained optimization). *Find the local extreme values and classify each as maxima, minima, or neither.*

a. $f(x_1, x_2) = 2x_1 - x_1^2 - x_2^2$

b. $f(x_1, x_2) = x_1^2 + 2x_2^2 - 4x_2$

c. $f(x_1, x_2) = x_1^3 - x_2^2 + 2x_2$

d. $f(x_1, x_2) = 4x_1 + 2x_2 - x_1^2 + x_1x_2 - x_2^2$

e. $f(x_1, x_2) = x_1^3 - 6x_1x_2 + x_2^3$

f. $f(x_1, x_2) = x_1 \sin x_2$

g. $f(x_1, x_2) = \frac{x_1}{1+x_1^2+x_2^2}$

Problem 3 (Constrained optimization I). *Solve the following problems. State the optimized value of the function at the solution.*

$$\begin{aligned}
&a. \quad \min_{x_1,x_2} x_1^2 + x_2^2 \; s.t. \; x_1x_2 = 1 \\
&b. \quad \min_{x_1,x_2} x_1x_2 \; s.t. \; x_1^2 + x_2^2 = 1 \\
&c. \quad \max_{x_1,x_2} x_1x_2^2 \; s.t. \; \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \\
&d. \quad \max_{x_1,x_2} x_1 + x_2 \; s.t. \; x_1^4 + x_2^4 = 1 \\
&e. \quad \max_{x_1,x_2,x_3} x_1x_2^2x_3^2 \; s.t. \; x_1 + x_2 + x_3 = 1
\end{aligned}$$

Problem 4 (Constrained optimization II). *A firm's inventory of a certain homogenous commodity, $I(t)$, is depleted at a constant rate per unit time $\frac{dI}{dt}$, and the firm reorders an amount x of the commodity, which is delivered immediately, whenever the level of inventory is zero. The annual requirement for the commodity is A, and the firm orders the commodity n times a year, where*

$$A = nx$$

The firm incurs two types of inventory costs: a holding cost and an ordering cost. The average stock of inventory is $\frac{x}{2}$, and the cost of holding one unit of the commodity is C_h, so $C_h \frac{x}{2}$ is the holding cost. The firm orders the commodity as stated above, n times a year, and the cost of placing one order is C_o, so $c_o n$ is the ordering cost. The total cost is then

$$C = C_h \frac{x}{2} + C_o n$$

a. In a diagram show how the inventory level varies over time. Prove that the average inventory level is $\frac{x}{2}$.

b. Minimize the cost of inventory, C, by choice of x and n subject to the constraint $A = nx$ using the Lagrange multiplier method. Find the optimal x as a function of the parameters C_o, C_h, and A. Interpret the Lagrange multiplier.

Problem 5 (Constrained optimization III). *Suppose x_1^* maximizes the function $f(x)$ over domain D_1 and x_2^* maximizes $f(x)$ over domain D_2. Is $x_1^* \in D_2$ necessary and sufficient for $f(x_1^*) \leq f(x_2^*)$? Prove or give a detailed counterexample.*

Problem 6 (Constrained optimization IV). *Eve consumes only bread and bottled water, and always prefers more of either to less. She has* \$21 *to spend each week. She has been shopping at Al's Food Emporium, where bread costs 5 cents a slice and water costs 10 cents a bottle. At these prices, she maximizes her utility by choosing 120 slices of bread and 150 bottles of water each week.*

Now suppose a competing grocer, Bob's Fooddrinkery, opens up across town. Bob charges 6 cents per slice of bread and 9 cents per bottle of water. That is, bread is more expensive and water is less expensive. Eve does not have time to go to both stores, so she must choose one. Would she be better off at Al's or Bob's?