Economics 590, fall 2007
Instructor: Jeremy Sandford

Homework 8
not collected

Problem 1 (Technology differences). *A social planner maximizes the utility of a representative agent, solving:*

$$\max_{\{c_t, I_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ subject to } \quad \begin{aligned} & c_t + I_t = f(k_t) \\ & k_t(1-\delta) + I_t = k_{t+1} \\ & \text{for } t = 0, 1, 2, ... \end{aligned}$$

where f is twice differentiable, increasing, and concave, $\lim_{c\to 0} u'(c) = \infty$, $\lim_{c\to\infty} u'(c) = 0$, $\lim_{k\to 0} f'(k) = \infty$, and $\lim_{k\to\infty} f'(k) = 0$.

a. Show that an increase in the depreciation rate δ decreases steady state consumption.

b. Show that more patient economies (higher β) have higher steady state levels of capital and output.

c. Suppose $f(k) = zg(k)$, where g is twice differentiable, increasing, concave, and satisfies $\lim_{k\to 0} g'(k) = \infty$, and $\lim_{k\to\infty} g'(k) = 0$. Think of z as a technology parameter. Show that higher levels of z result in higher steady state consumption.

d. Assume that $zg(k) = zk^{.33}$. Describe what range of values of the technology parameter z would be required to make the model consistent with differences in per capita income across countries that are as high as 30 to 1. (nb. the level of technology for the "poorest" country can be chosen equal to 1.)

Problem 2 (Productive labor). *Consider the basic economy as described in problem 1, with one modification. Suppose that each agent has one unit of time each period, which he can allocate to labor, which is productive, or leisure, which is enjoyable. Call the amount of labor at time t n_t and the amount of leisure $1 - n_t$. The social planner then solves*

$$\max_{\{c_t, I_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t) \text{ subject to } \quad \begin{aligned} & c_t + I_t = f(k_t, n_t) \\ & k_t(1-\delta) + I_t = k_{t+1} \\ & \text{for } t = 0, 1, 2, ... \end{aligned}$$

where u and f are strictly increasing in each argument, concave, and twice differentiable. In addition, $\lim_{k\to 0} \frac{\partial}{\partial k} f(k,n) = \infty$, $\lim_{n\to 0} \frac{\partial}{\partial n} f(k,n) = \infty$, $\lim_{k\to\infty} \frac{\partial}{\partial k} f(k,n) = 0$, $\lim_{n\to\infty} \frac{\partial}{\partial n} f(k,n) = 0$.

a. Describe the steady state of this economy. If necessary, make additional assumptions to guarantee that it exists and is unique.

b. What is the impact, if any, on the steady state level of employment (n^*) of an increase in productivity z? Are there cases in which "high technology" countries (high z) have lower employment levels (more leisure)?

Problem 3 (Government spending). *A social planner wishes to maximize the utility of a representative agent, but he is constrained to allocate an exogenous amount g to the government each period. Thus, he solves:*

$$\max_{\{c_t, I_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ subject to } \quad c_t + I_t + g = f(k_t)$$
$$k_t(1-\delta) + I_t = k_{t+1}$$
$$\text{for } t = 0, 1, 2, ...$$

where f is twice differentiable, increasing, and concave, $\lim_{c\to 0} u'(c) = \infty$, $\lim_{c\to\infty} u'(c) = 0$, $\lim_{k\to 0} f'(k) = \infty$, *and* $\lim_{k\to\infty} f'(k) = 0$.

a. Write down the Lagrangian associated with this problem

b. Take the first order conditions of said Lagrangian; solve for the Euler equation which results.

c. In class we discussed the existence and uniqueness of a steady state when $g = 0$. *Following this, argue that a steady state exists and is unique if and only if g is below some cutoff level of spending, say* $\bar{g}$.

d. Suppose the planner were allowed to choose government spending, i.e. to choose the sequence $\{g_t\}$. *What would the chosen sequence look like?*

Problem 4 (Hyperbolic discounting). *Economists say that "hyperbolic discounting" describes the behavior of someone saying 'today I'll spend, and tomorrow I'll save', 'I'll eat a big meal today, and cut back tomorrow', or 'today I'll just watch football, tomorrow I'll really start that exercise program', day after day. Such attitudes are not time consistent, because when tomorrow rolls around, the person says the same thing. This type of model has been used to explain the behavior of smokers and the low US savings rate (see the papers of David Laibson).*

Consider the following model of hyperbolic discounting. For simplicity, assume full depreciation ($\delta = 1$), *log utility, and Cobb-Douglas production. The planner thus solves:*

$$\max_{\{c_t, I_t, k_{t+1}\}_{t=0}^{\infty}} \quad log(c_0) + \gamma\left[\beta log(c_1) + \beta^2 log(c_2) + \beta^3 log(c_3) + ...\right]$$
$$\text{subject to } c_t + k_{t+1} = k_t^{\alpha}$$
$$\text{for } t = 0, 1, 2, ...$$

where $\gamma \in (0, 1)$.

a. Write down a value function $V[k]$ *corresponding to the above problem (nb. this is a little tricky. Look closely at the standard example we did in class on 11/29 and think about it carefully).*

b. Set the derivative of $V[k]$ *with respect to* k' *to 0, and use this condition to determine the transition rule* $k'(k)$ *and consumption* $c(k)$.

c. Derive a closed form solution for $V[k]$, *using the guess-and-verify approach (again, look carefully at the standard model studied in the 11/29 class).*

d. Define the savings rate to be $\frac{k'}{k^{\alpha}}$ *to be the fraction of all output saved for the subsequent period. Does this savings rate change over time? Show that the actual savings rate in period t is different from that which was planned from the perspective of period* $t-1$.

Problem 5 (Permanent income). *Mose earns income w in every period; he also has a bank account that earns interest r. Denote his current consumption by c and current wealth by x. His budget constraint is thus*

$$x' + c = x(1 + r) + w$$

Mose wants to maximize his lifetime utility $\sum_{t=0}^{\infty} \beta^t u(c_t)$. Assume, for this problem only, that $\beta(1 + r) = 1$.

a. Write down a value function $V[x]$ describing Mose's utility-maximization problem.

b. Solve for optimal one-period-ahead wealth, x', as a function of x.

c. Let $u(c) = c$. Give a closed-form solution for the value function $V[x]$.

Problem 6 (Guess and verify). *A vintner[1] has one unit of labor to use each day. He can allocate that labor between the making of bread and the pressing of grapes for grape juice. The bread he makes today he can consume today. The grape juice he makes today will become tomorrow's wine (he doesn't care for grape juice). The production technology is linear: it produces one unit of bread per unit of labor allocated to baking, and one unit of juice per unit of labor allocated to grape pressing, and one unit of wine per unit of grape juice left to ferment. The transformation of juice into wine requires no labor, only time. The vintner allocates his labor so as to maximize the utility of his own consumption. His utility function has the form:*

$$\sum_{t=0}^{\infty} \beta^t \sqrt{b_t w_t}$$

where b_t and w_t are the bread and wine consumption, respectively, in period t. The initial wine consumption w_0 is given. The discount factor is $\beta \in (0, 1)$.

a. Write down the value function associated with this problem.

b. Guess that the value function has the form $V(w) = \alpha\sqrt{\gamma + w}$, where α and γ are unknown parameters, and go as far as you can in verifying that V has this form. Solve for parameters α and γ.

c. State the optimal choice of next period's wine, w', as a function of current wine w.

[1] vintner=one who makes wine