Homework 10

due 12/1/08

Problem 1 (Finite-horizon dynamic programming I) (Sundaram, pg 278)

Solve parts a, b, and c of problem 1 on page 278 of the Sundaram book.

a. this is a generalization of what we did in class the week before Thanksgiving.

b. Here, marginal utility of consumption is constant, and so it does not matter how it is allocated across periods; there are many solutions

c. With increasing marginal utility, the optimal solution is to maximize consumption in one period and consume zero in all other periods. It does not matter which period, so there are T solutions.

Problem 1 (Finite-horizon dynamic programming II) (Sundaram pg 278)

Consider the problem of optimal harvesting of a natural resource. A firm (say, a fishery) begins with a given stock of $y > 0$ of a natural resource (fish). In each period $t = 1, 2, ..., T$ of a finite horizon, the firm must decide how much of the resource to sell on the market that period. If the firm decides to sell x units of the resource, it receives a profit of $\log(x)$. The amount $(y - x)$ of the resource left unharvested grows to an available amount of $(y - x)^\alpha$ at the beginning of the next period. The firm wishes to choose a strategy that will maximize the sum of its profits over the model's T-period horizon.

Solve this problem for the firm's optimal choices of x and y over periods $0, 1, ..., T$. Prove your answers.[1]

This is a trivial extension of material covered in lecture

Problem 3 (Finite-horizon dynamic programming III) (Sundaram pg 279)

Let an $m \times n$ matrix A be given. Consider the problem of finding a path between entries a_{ij} in the matrix A which (i) starts at a_{11} and ends at a_{mn}, (ii) which moves only to the right or down, and (iii) which maximizes the sum of the entries a_{ij} encountered. Express this as a dynamic programming problem. Using backwards induction, solve the problem when the matrix A is given by

$$\begin{bmatrix} 4 & 9 & 3 & 6 & 3 \\ 5 & 6 & 6 & 4 & 4 \\ 6 & 7 & 1 & 1 & 0 \\ 4 & 3 & 5 & 1 & 9 \end{bmatrix}$$

Let $V(m,n)$ be the optimized sum over the path starting at a_{mn} and ending at a_{55}, moving only right or down. Trivially, $V(4,5) = 9$, $V(3,5) = 0 + V(4,5) = 9$, $V(2,5) = 4 + V(3,5) = 13$, $V(1,5) = 3 + V(2,5) = 16$, $V(4,4) = 1 + V(5,5) = 10$, $V(4,3) = 5 + V(4,4) = 15$, $V(4,2) = 3 + V(3,4) = 18$, $V(4,1) = 4 + V(4,2) = 22$.

Then, $V(3,4) = 1 + \max\{V(5,4), V(4,5)\} = 11$, $V(2,4) = 4 + \max\{V(2,5), V(3,4)\} = 17$, $V(1,4) = 6 + \max\{V(2,4), V(1,5)\} = 23$, $V(3,3) = 1 + \max\{V(4,3), V(3,4)\} = 16$, $V(3,2) = 7 + \max\{V(4,2), V(3,3)\} = 25$, $V(3,1) = 6 + \max\{V(3,2), V(4,1)\} = 31$.

Then, $V(2,3) = 6 + \max\{V(3,3), V(2,4)\} = 23$, $V(1,3) = 3 + \max\{V(2,3), V(1,4)\} = 26$, $V(2,2) = 6 + \max\{V(2,3), V(3,2)\} = 31$, $V(2,1) = 5 + \max\{V(2,2), V(3,1)\} = 36$.

[1] Much of this was answered in the 11/18 lecture. What remains is to rigorously demonstrate what x and y are in period t, for generic t.

Finally, $V(1,2) = 9 + \max\{V(2,2), V(1,3)\} = 40$, $V(1,1) = 4 + \max\{V(2,1), V(1,2)\} = 44$.

Thus, the optimal path is $(1,1),(1,2),(2,2),(3,2),(4,2),(4,3),(4,4),(4,5)$. Note that this can be solved easily by, for example, writing the value to starting at each entry in red superscript above the entry, and working backwards. I use the language of value functions to emphasize the dynamic programming aspect of the problem.