Homework 1

some answers

Problem 1 (Set relationships) For each of the following, determine whether $A = B$. If not, state whether $A \subset B$, $B \subset A$, or neither. Prove your answers.

a. $A = \{0\}$, $B = \bigcap_{n=1}^{\infty} \Gamma_n$, where $\Gamma_n = [0, \frac{2}{n}]$ for $n = 1, 2, 3, ...$

b. $A = (0, 1)$, $B = \bigcup_{n=1}^{\infty} \Gamma_n$, where $\Gamma_n = (\frac{1}{n+1}, \frac{1}{n})$, for $n = 1, 2, 3, ...$

c. $A = (\Gamma_1 \times \Gamma_2) \bigcup (\Gamma_3 \times \Gamma_4)$, $B = (\Gamma_1 \bigcup \Gamma_3) \times (\Gamma_2 \bigcup \Gamma_4)$, where $\Gamma_n \subset \mathbb{R}$ for $n \in \{1, 2, 3, 4\}$.

d. $A = \{x \in \mathbb{R} : x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}\}$, $B = \{x \in \mathbb{R} : x = \frac{a}{b}$ for some $a, b \in \mathbb{Z},$ and b is an even number$\}$

e. $A = \{x \in \mathbb{R} : x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}\}$, $B = \{x \in \mathbb{R} : x = \frac{a}{b}$ for some $a, b \in \mathbb{Z},$ and b is an odd number$\}$

Problem 2 (Convex sets I) Suppose A and B are convex sets in $\mathbb{R}^n$. Show that $A \cap B$ is also convex.

Consider $x_1, x_2 \in A \cap B$. That x_1 and x_2 are in $A \cap B$ implies that both points are in A. As A is convex, we know that any convex combination of two of its points $\alpha x_1 + (1 - \alpha)x_2$ is itself in A. Similarly, that x_1 and x_2 are in $A \cap B$ implies that both points are in B; as B is convex, it must be that $\alpha x_1 + (1 - \alpha)x_2 \in B$ for any $\alpha \in (0, 1)$. As any convex combination of points in $A \cap B$ is in both A and B, it certainly is also in $A \cap B$, and thus $A \cap B$ is convex.

Problem 3 (Convex sets II) Let A and B be convex sets. Show (by counterexample) that $A \cup B$ need not be convex. Examples abound. For example, take $A = [0, 1]$ and $B = [2, 3]$.

Problem 4 (Mappings I) Let $A = (0, 10)$ and $B = \mathbb{R}$.

a. Give an example of a mapping from A to B which is neither one-to-one nor onto.

Consider $f : A \to B$, with $f(x) = 7$

b. Give an example of a mapping from A to B which is one-to-one but not onto.

Consider $f : A \to B$, with $f(x) = x$

c. Give an example of a mapping from A to B which is bijective.

Consider $f : A \to B$, with $f(x) = -\frac{1}{x} + \frac{1}{10-x}$

d. Give an example of a mapping from A to B which is onto but not one-to-one (note: for part d only, you may draw a picture for your answer).

Problem 5 (De Morgan's laws) In class, we showed that for any sets $A_1, A_2, A_3, ..., A_n$,

$$(\bigcup_{i=1}^{n} A_i)^c = \bigcap_{i=1}^{n} A_i^c$$

This is also know as the first of two De Morgan's laws, named for the mathematician and logician Augustus De Morgan (1806-1871). The second of De Morgan's laws says that

$$(\bigcap_{i=1}^{n} A_i)^c = \bigcup_{i=1}^{n} A_i^c$$

Prove the second of De Morgan's laws, (hint: imitate and/or use the proof of the first law from class).

Problem 6 (Mappings II) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by the rule $f(x) = x^3 - x$.

Prove a-c to be true or false

a. f is one-to-one

False. $f(0) = f(1) = 0$

b. f is onto

True. To prove this, we must show that for any $y \in \mathbb{R}$, there exists some $x \in \mathbb{R}$ such that $x^3 - x = y$. First, consider a generic $y > 0$. Note that $f(1) = 0$, and that $f(x)$ is increasing and continuous for $x \geq 1$ (for example, it has a positive first derivative). As it is also convex (positive second derivative), it is not bounded from above, so f increases from 0 to ∞ over the interval $[1, \infty)$. An identical story gives us that f takes on values between $-\infty$ and 0 over the interval $(-\infty, -1]$. Thus, the range of f is equal to the codomain, or $f(\mathbb{R}) = \mathbb{R}$, and so f is surjective.

c. f is bijective

d. If you argued statement c to be false (hint: you should have), restrict the domain and/or range of f so that you get a new function $g : A \to B$ where $g(x) = x^3 - x$, $A \subset \mathbb{R}$, and $B \subset \mathbb{R}$ such that g is bijective. Graph g and g^{-1}. Note that there are many possible choices of g.

Problem 6 (Mappings III) Consider sets A and B, each having a finite number of elements. That is, $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_m\}$, for some integers m and n.

Prove each of the following statements to be either true or false:

a. If $m < n$, there exists no function $g : A \to B$ that is one-to-one.

True. To see this, suppose g were injective. Then, necessarily, $g(a_1) \neq g(a_2) \neq ... \neq g(a_m)$. Given that there are only m values in the codomain, it would have to be that each $b \in B$ would be equal to $g(a_j)$ for some $j \in \{1, 2, ..., m\}$. But then $g(a_{m+1}) \in B$ as well, and so it must be that $g(a_{m+1}) = g(a_j)$, for some j, a contradiction.

b. If $m < n$, every function $g : A \to B$ is onto.

False. Consider $g(x) = b_1$.

c. If $m = n$, every function $g : A \to B$ is one-to-one.

False. Consider $g(x) = b_1$.

d. If $m > n$, there exists a function $g : A \to B$ which is one-to-one.

True. Consider $g(x_i) = y_i$ for $i = 1, 2, ..., n$.