Homework 2

due 9/15/08

Problem 1 (Images, preimages) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = 3x + 2y$ for $(x, y) \in \mathbb{R}^2$.

a. Find the image under f of the set $[0, 5] \times [3, 7]$, or $f([0, 5] \times [3, 7])$.[6,29]

b. Find the preimage under f of the set $\{10\}$, that is $f^{-1}(\{10\})$. All points on the line $y = 5 - \frac{3}{2}x$

c. Is f one-to-one? Onto? Prove your claims. F is surjective, but not injective. To show that it is surjective, for any $z \in \mathbb{R}, z = f(x, y)$ if $y = \frac{z}{2} - \frac{3}{2}x$. To show that it is not injective, note that $f(1, 5) = f(3, 2) = 13$.

Problem 2 (Cartesian products) (Sundaram, page 69 #30) Given two subsets A and B of $\mathbb{R}$, recall that their *Cartesian product* $A \times B \subset \mathbb{R}^2$ is defined as

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Give an example of a set $X \subset \mathbb{R}^2$ that *cannot* be expressed as the Cartesian product of sets $A, B \subset \mathbb{R}$.Consider $(0, 0) \cup (3, 1)$

Problem 3 (Orders I)[1] Consider the (preference) ordering of a 2-dimensional consumption set ($\mathbb{R}^2_+$) given by $(x_1, x_2) \succ (y_1, y_2) \iff x_1 * x_2 > y_1 * y_2$, and $(x_1, x_2) \sim (y_1, y_2) \iff x_1 * x_2 = y_1 * y_2$, for all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2_+$.

a. Show that this ordering is transitive, that is that $x \succ y$ and $y \succ z$ implies $x \succ z$ for all $x, y, z \in \mathbb{R}^2_+$.

b. Show that this ordering is *strongly monotone*, that is that $x_1 \geq y_1$ and $x_2 \geq y_2$ with $x \neq y$ implies $x \succ y$.

c. Show that the ordering is convex, that is for all $x \in \mathbb{R}^2_+$, the set $\{y \in \mathbb{R}^2_+ : y \succsim x\}$ is a convex set.

d. Give an example of a (preference) ordering on $\mathbb{R}^2_+$ which is not strongly monotone. Give another example of an ordering which is not convex.

Problem 4 (Orders II) Consider the lexicographic ordering on $\mathbb{R}^2_+$ discussed in class, that is $(x_1, x_2) \succ (y_1, y_2)$ if "$x_1 > y_1$" or "$x_1 = y_1$ and $x_2 > y_2$" and $(x_1, x_2) \sim (y_1, y_2)$ if "$x_1 = y_1$ and $x_2 = y_2$".

a. Show that this order is both strongly monotone and convex.

b. In question 3, it is clear that the preferences given can be represented by the utility function $u(x) = x_1 * x_2$, i.e. that $x \succsim y \iff u(x) \geq u(y)$. Go as far as you can in showing that lexicographic preferences cannot be represented by any utility function.

Problem 5 (Metrics I) For $x, y \in \mathbb{R}^n$, consider the distance function given by

$$\begin{aligned} d(x, y) &= 1 \text{ if } x \neq y \\ &\quad 0 \text{ if } x = y \end{aligned}$$

Prove that $d(\cdot)$ is a metric on $\mathbb{R}^n$.

[1]When discussing preference orderings, it is common to use the notation $\succ$, $\sim$, and $\succsim$ in place of $>$, $=$, and $\geq$.

Problem 6 (Metrics II) (Rudin, page 44, #11) For $x \in \mathbb{R}$ and $y \in \mathbb{R}$, define

$$\begin{aligned} d_1(x,y) &= (x-y)^2 \text{This fails the triangle inequality. For example, if } x=0,\ y=5, \text{ and } z=10,\ (x-z)^2 > (x-y)^2+(y-z)^2 \\ d_2(x,y) &= \sqrt{|x-y|} \text{This is a metric. Triangle inequality follows from the triangle inequality for } |\cdot| \\ & \quad \text{and the fact that } \sqrt{\cdot} \text{ is an increasing function over } \mathbb{R}_+ \text{ and so is order-preserving.} \\ d_3(x,y) &= |x^2-y^2| \text{This is not a metric. Consider } x=2, y=-2. \text{ Then, } d_3(x,y)=0, \text{ even though } x \neq y. \\ d_4(x,y) &= |x-2y| \text{This is not a metric. Consider } x=1, y=\frac{1}{2};\ |x-2y|=0, \text{ so this assigns zero distance to distinct points.} \end{aligned} \quad (1)$$

Determine whether or not each of these is a valid metric.

Problem 7 (Metrics III) Is every metric on $\mathbb{R}^2$ order-preserving? That is, if $d_1 : \mathbb{R}^2 \to \mathbb{R}$ and $d_2 : \mathbb{R}^2 \to \mathbb{R}$ are both valid metrics on $\mathbb{R}^2$, is it necessarily the case that, for any $x, y, z \in \mathbb{R}^2$,

$$d_1(x,y) < d_1(x,z) \Rightarrow d_2(x,y) < d_2(x,z)?$$

The taxicab metric, with the horizontal distance weighted differently than the vertical distance, evidently does not necessarily produce the same orders as the normal Euclidean metric; see the example I did in class.

Problem 8 (Metrics IV) Google Maps provides a distance in miles between two addresses. Indeed, the program allows you to input locations in terms of latitude and longitude, so Google Maps could reasonably be said to map any two points in Lexington to a real number representing the distance between those two points. Ignoring technical problems with the website, could the Google Maps algorithm be said to define a metric over the set of all points in Lexington? As Lexington has many one-way streets, it is not the case that $d(x,y) = d(y,x)$ for every x and y. For example, getting from Limestone and Euclid to Limestone and Vine does not take as long as getting from Limestone and Vine to Limestone and Euclid, as the latter requires continuing north on Limestone to Main, turning left, turning left onto Upper, heading south past Euclid, making a U-turn across from Administration Drive, and heading back north on Limestone to get to Euclid.