Homework 3

due 9/22/08

Problem 1 (Closed sets) Consider $\mathbb{R}^n$ with the Euclidean metric. Prove that $F \subset \mathbb{R}^n$ is closed if and only if the complement of F is open.

Problem 2 (Intersections of open sets) In $\mathbb{R}^n$ with the Euclidean metric,

a. Prove that the intersection of any finite number of open sets is an open set.

b. Give an example of an infinite (countable or uncountable) collection of open sets such that the intersection is not open. Be as explicit as possible.

c. Give an example of an infinite (countable or uncountable) collection of closed sets such that the union is not closed. Again, be very explicit about what the union is and how you know it is not closed.

Problem 3 (Closed sets II) Consider $\mathbb{R}^n$ with the Euclidean metric. Prove the following statement:

$F \subset \mathbb{R}^n$ is closed if and only if for every sequence $\{x_n\}$ contained in F,

$$\lim_{n\to\infty} x_n = x \quad \Rightarrow \quad x \in F. \tag{1}$$

Again, use the definition of 'closed' given in class.

Problem 4 (Extreme values) (Sundaram page 68, #16) Find the supremum, infimum, maximum, and minimum, if they exist, for the following sets:

a. $A_1 = \{x \in [0,1] : x \text{ is irrational}\}$

b. $A_2 = \{x : x = \frac{1}{n}, \text{ for } n = 1, 2, ...\}$

c. $A_3 = \{x : x = 1 - \frac{a}{n}, \text{ for } n = 1, 2, ...\}$ (note: take a to be some real number in $(0, 1)$)

d. $A_4 = \{x \in [0, \pi] : sin(x) > \frac{1}{2}\}$

e. $A_5 = \phi$, the empty set

Problem 5 (Convex sets) A set A is convex if, for every $x_1, x_2 \in A$, and for every $\alpha \in (0, 1)$,

$$\alpha x_1 + (1 - \alpha)x_2 \in A \tag{2}$$

(this is the usual definition from class).

If a set B satisfies $\frac{1}{2}x_1 + \frac{1}{2}x_2 \in B$ for all $x_1, x_2 \in B$, does it follow that B is convex?

Problem 6 (Metric spaces and open sets) Fact: sets are open and closed *relative to the metric space in which they are contained*. That is, a set which is open in metric space $(A, d_1(\cdot))$ may not be open when seen as a subset of $(B, d_2(\cdot))$.

Demonstrate your understanding of this by arguing that the set $(0, 5)$ is open when seen as a subset of $(\mathbb{R}, |\cdot|)$, but not open as a subset of $(\mathbb{R}^2, |\cdot|)$, where $|\cdot|$ is the standard Euclidean metric.

Problem 7 (Rational numbers) $\mathbb{Q} = \{x \in \mathbb{R} : x = \frac{a}{b}, \text{ for integers } a, b\}$ denotes the set of rational numbers. Is $\mathbb{Q}$ an open subset of the Euclidean space $(\mathbb{R}, |\cdot|)$, where $|\cdot|$ is the Euclidean metric? Is $\mathbb{Q}$ a closed subset of the same?

Problem 8 (Open covers) An *open cover* of a set $A \subset \mathbb{R}^n$ is a collection of open sets $\{O_i\}_{i \in I}$, $O_i \subset \mathbb{R}^n$ for each $i \in I$, such that $A \subset \cup_{i \in I} O_1$.

a. Go as far as you can in proving that every open cover of the interval $[0,1] \subset \mathbb{R}$ has a finite subcover, that is that for any sets $\{O_i\}_{i \in I}$ such that $[0,1] \subset \cup_{i \in I} O_i$, there exist n elements of $\{O_i\}_{i \in I}$, call them $O_{i(1)}, O_{i(2)}, ..., O_{i(n)}$, such that $[0,1] \subset \cup_{j=1}^{n} O_{i(j)}$.

b. Give an example of an open cover of $(0,1)$ which has no finite subcover.