Homework 4

answers

Problem 1 (Weierstrass theorem) (Sundaram, page 98 #13) A monopolist faces a downward sloping inverse-demand curve $p(x)$ that satisfies $p(0) < \infty$ and $p(x) \geq 0$ for all $x \in \mathbb{R}_+$. The cost of producing x units is given by $c(x) \geq 0$, where $c(0) = 0$. Suppose $p(\cdot)$ and $c(\cdot)$ are both continuous on $\mathbb{R}_+$. The monopolist wishes to maximize profit, $\pi(x) = xp(x) - c(x)$, subject to the constraint $x \geq 0$.

a) Suppose there is $x^* > 0$ such that $p(x^*) = 0$. Show that the Weierstrass theorem can be used to prove the existence of a solution to this problem. As p is downward-sloping, $p(x) \leq 0$ for $x \geq x^*$. In this range, $\pi(x) = xp(x) - c(x) \leq xp(x) \leq 0$, where the first inequality follows because $c(x) \geq 0$ and the second because $p(x) \leq 0$ for $x > x^*$. Clearly, $\pi(0) = 0$, and so we can conclude that the maximizer of π over $[0, \infty)$ is not in the interval (x^*, ∞). This leaves the interval $[0, x^*]$, a closed and bounded interval; the continuous function π thus obtains a maximum on $[0, x^*]$ by the Weierstrass theorem, and as this maximum is surely greater than any value π takes on over (x^*, ∞), we can conclude that a maximizer of π exists on $[0, \infty)$.

b) Now suppose instead there is $\tilde{x} > 0$ such that $c(x) \geq xp(x)$ for all $x \geq \tilde{x}$. Show, once again, that the Weierstrass theorem can be used to prove existence of a solution.A similar argument to that of 1 supplies that $\pi(0) \geq \pi(x)$ for all $x \geq \tilde{x}$, and as the continuous function π obtains a maximum on the closed and bounded interval $[0, \tilde{x}]$, we can conclude that π has a maximum on $[0, \infty)$.

c) What about the case where $p(x) = \bar{p}$ for all x (the demand curve is infinitely elastic) and $c(x) \to \infty$ as $x \to \infty$?That $c(x) \to \infty$ ensures that for some $\hat{x}$ we have that $c(x) > xp(x)$ for $x \geq \hat{x}$. The proof of part b) then applies.

Problem 2 (Weierstrass theorem II) (Sundaram, page 97 #2) Suppose $A \subset \mathbb{R}^n$ is a set consisting of a finite number of points $\{x_1, x_2, ..., x_p\}$. Show that any function $f : A \to \mathbb{R}$ has a maximum and a minimum on A. Is this result implied by the Weierstrass theorem? Explain.
textsfThe function f takes on values $f(x_1), f(x_2), ..., f(x_n)$; this finite list necessarily has a largest member, and so f obtains a maximum on A. This is not immediately implied by the Weierstrass theorem as f is not continuous on A.

Problem 3 (Weierstrass theorem III) (Sundaram, page 97 #1) Prove or counter the following statement:

If f is a continuous real-valued function on a bounded (but not necessarily closed) set A, then $\sup f(A)$ is finite. (nb. $\sup f(A) = \sup\{y \in \mathbb{R} : y = f(x) \text{ for some } x \in A\}$).This is false; consider the interval $(0, 1)$ and the function $f : (0, 1) \to \mathbb{R}$ described by $f(x) = \frac{1}{1-x}$. This is continuous on the domain, but goes off to infinity as x approaches 1, and so there does not exist a finite $\sup f(x)$ on $(0, 1)$.

Problem 4 (Sequences) (Sundaram, page 67 #3) Let $\{x_n\}$, $\{y_n\}$ be sequences in $\mathbb{R}^n$ such that $x_n \to x$ and $y_n \to y$. For each n, let $z_n = x_n + y_n$, and let $w_n = x_n * y_n$. Show that $z_n \to (x + y)$ and $w_n \to x * y$. Since $x_n \to x$, we know that for any $\epsilon > 0$ there is some N_ϵ such that $n > N_\epsilon \Rightarrow |x_n - x| < \epsilon$. Likewise, that $y_n \to y$, we know that for any $\gamma > 0$ there is some N_γ such that $n > N_\gamma \Rightarrow |y_n - y| < \gamma$. Now, for any $\delta > 0$, set $\epsilon = \gamma = \frac{\delta}{2}$. For $n > \max\{N_\epsilon, N_\gamma\}$, $|z_n - z| = |x_n + y_n - x - y| < |x_n - x| + |y_n - y| < \frac{\delta}{2} + \frac{\delta}{2} = \delta$, where the first inequality is the triangle inequality. We have thus shown that for any $\delta > 0$, there exists N_δ such that $|z_n - z| < \delta$, and thus that $z_n \to z$. A similar proof should work for w_n.

Problem 5 (Sequences II) In $\mathbb{R}^n$ with metric $d(x, y)$, a sequence $\{x_n\}$ is called a *Cauchy sequence* if, for any $\epsilon > 0$, there exists a number $N(\epsilon)$ such that $n, m > N(\epsilon)$ implies that $d(x_n, x_m) < \epsilon$.

Prove that any convergent sequence in $\mathbb{R}^n$ is a Cauchy sequence.

If a sequence x_n converges to a point x, than, for any $\delta > 0$ there exists $N_\delta > 0$ such that $n > N$ implies that $d(x_n, x) < \delta$. Pick $\delta = \frac{\epsilon}{2}$, and note that, for $n, m > N_\delta$, $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, with the first inequality being the triangle inequality.

Problem 6 (Metric spaces) Prove or counter each of the following statements:

a. If A is a non-empty closed subset of $\mathbb{R}^n$, and $x \notin A$, there is a point in A that is nearest to x, under the metric $d(x, y) = \sqrt{\Sigma_{i=1}^n (x_i - y_i)^2}$. Pick any point $y \in A$, and then define $B = \{z \in \mathbb{R}^2 : d(z, x) \leq d(y, x)\}$ to be the set of all points in $\mathbb{R}^2$ within distance $d(x, y)$ of point x, i.e. at least as close to x as y is. The closest point to x in A is surely in B, as y is in both A and B. Now, B is simply a circle containing its boundary, and so is closed. Intersections of closed sets are closed, and so $A \cap B$ is closed, and is clearly bounded. $d(x, y)$ is continuous in y and thus obtains a minimum in y on $A \cap B$ by the Weierstrass theorem, and so there is a point in A which is closest to x.

b. If A is a non-empty open subset of $\mathbb{R}^n$, and $x \notin A$, there is a point in A that is nearest to x, under the metric $d(x, y) = \sqrt{\Sigma_{i=1}^n (x_i - y_i)^2}$. This is false; suppose $n = 1$, and let $A = (0, 1)$. Consider the point $x = 7$. There is no point in A which is closest to 7 (a simple proof will show this).

c. If A is a non-empty closed and bounded subset of $\mathbb{R}^n$, and $x \notin A$, there is a unique point in A that is nearest to x, under the metric $d(x, y) = \sqrt{\Sigma_{i=1}^n (x_i - y_i)^2}$. The point need not be unique. Suppose $A = \{(1, 0)\} \cup \{(-1, 0)\}$, and $x = (0, 0)$. Both points in A are then equidistant from x.

Problem 7 (Basic optimization) Prove or counter the following statement:

If $g : \mathbb{R} \to \mathbb{R}$ is a function (not necessarily continuous) which has a maximum and minimum on $\mathbb{R}$, and if $f : \mathbb{R} \to \mathbb{R}$ is continuous, $h(x) = f(g(x))$ necessarily has a maximum on $\mathbb{R}$. False. Consider

$$g(x) = \begin{cases} -1, & \text{if } x < -1 \\ x, & \text{if } x \in (-1, 0) \\ -1, & \text{if } x \in [0, 1) \\ 1, & \text{if } x > 1 \end{cases}$$

and $f(x) = -x^2$. Then, $h(x) = f(g(x))$ has no maximum. To see this, $h(x) = -1$ for all x outside of the interval $(-1, 0)$, and $h(x) > -1$ for all $x \in (-1, 0)$. But, for any $x \in (-1, 0)$, $h(\frac{1}{2}x) > h(x)$, and so no maximum can exist in this interval.