Homework 7

due 11/3/08

Problem 1 (Constrained optimization I) (Sundaram page 142) Consider the problem

$$\min x^2 + y^2 \text{ subject to } (x-1)^3 = y^2$$

a. Solve this minimization problem.

b. Show that the Lagrange multiplier method does not work in this case. Can you explain why?

See 11/6/08 lecture.

Problem 2 (Constrained optimization II) Find the maximum of the function $f(x, y, z) = xyz$ subject to the constraints $x \geq 0$, $y \geq 0$, $z \geq 0$, and $xy + yz + xz = 2$. The constraint set is compact, and the objective function is continuous, so a solution exists by the Weierstrass theorem. Apply the theorem of Lagrange to maximize the objective function over $\mathbb{R}^3_{++} \bigcap \{(x, y, z) : xy + yz + xz - 2 = 0\}$. Note that the derivative of $xy + yz + xz - 2$ is zero only at the origin, which is not in the constraint set, so the constraint qualification is met. Moreover, the maximizer is certainly interior (i.e. has $x > 0, y > 0, z > 0$), as otherwise the objective function is 0, less than the objective function at any feasible interior point. Thus, the problem's solution satisfies the following for some λ:

$$\begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} = \lambda \begin{pmatrix} y+z \\ x+z \\ x+y \end{pmatrix}$$
$$xy + yz + xz = 2$$

This has exactly one critical point, at $x = y = z = \frac{\sqrt{2}}{\sqrt{3}}$. Conclude that this point maximizes the objective function over the constraint set.

Problem 3 (Constrained optimization II) Answer each of the following:

a. Find the maximum and minimum distance from the origin to the ellipse $x^2 + xy + y^2 = 3$.

Both the max and the min exist by the Weierstrass theorem. Apply the theorem of Lagrange to maximize/minimize $x^2 + y^2$ over $\mathbb{R}^2 \bigcap \{(x, y) : x^2 + xy + y^2 - 3 = 0\}$. Both max and mix appear as points satisfying

$$\begin{pmatrix} 2x \\ 2y \end{pmatrix} = \lambda \begin{pmatrix} 2x+y \\ 2y+x \end{pmatrix} \tag{1}$$
$$x^2 + xy + y^2 = 3 \tag{2}$$

for some λ. The solutions to 1 lie along the lines $y = x$ and $y = -x$. Each of these lines intercept the constraint set twice. We thus have four critical points, $(1, 1), (-1, -1), (\sqrt{3}, -\sqrt{3}), (-\sqrt{3}, \sqrt{3})$. Evaluating the objective function at each of these four points gives us that the first two are minima and the latter two are maxima.

b. Find the point on the parabola $y = x^2$ which is closest to the point $(2, 1)$.

c. Find the point closest to the origin in $\mathbb{R}^3$ which is on both the plane $3x + y + z = 5$ and $x + y + z = 1$.

If no maximum or minimum exists, prove it.

Problem 4 (Applied optimization I) Your ship is overdue in port, and the beer is running out. The remaining supplies are divided up and you get 22.5 liters. The ship will not reach port before tomorrow morning, and there is a 60% chance that it will arrive then. You can't take beer with you when you leave the ship, so you could drink it all today, to make sure it isn't wasted. On the other hand, there is a 40% chance that you will still be afloat all day tomorrow, and a 10% chance that you will be afloat the day after that. You could save some beer in case you need it for the second day, or the third. It is certain that you will reach port before the fourth day.

You are an expected utility maximizer, and your utility function is $U(B) = 6000B - 250B^2$, where B is daily beer consumption.[1] How much beer should you drink today, and how much should you save for day 2? For day 3?

$B_1 = 11, B_2 = 9.5, B_3 = 2.$

Problem 5 (Applied optimization II) (Sundaram page 143)

A firm's inventory of a certain homogenous commodity, $I(t)$, is depleted at a constant rate per unit time $\frac{dI}{dt}$, and the firm reorders an amount x of the commodity, which is delivered immediately, whenever the level of inventory is zero. The annual requirement for the commodity is A, and the firm orders the commodity n times a year, where

$$A = nx$$

The firm incurs two types of inventory costs: a *holding cost* and an *ordering cost*. The average stock of inventory is $\frac{x}{2}$, and the cost of holding one unit of the commodity is C_h, so $C_h\frac{x}{2}$ is the holding cost. The firm orders the commodity as stated above, n times a year, and the cost of placing one order is C_o, so c_on is the ordering cost. The total cost is then

$$C = C_h\frac{x}{2} + C_on$$

a. In a diagram show how the inventory level varies over time. Prove that the average inventory level is $\frac{x}{2}$.

b. Minimize the cost of inventory, C, by choice of x and n subject to the constraint $A = nx$ using the Lagrange multiplier method. Find the optimal x as a function of the parameters C_o, C_h, and A. Interpret the Lagrange multiplier.

[1]This means that if you were to consume B_1 beer on day 1, B_2 on day 2, and B_3 on day 3, your total utility would be $U(B_1) + .4U(B_2) + .1U(B_3)$