Homework 8

due 11/10/08

Problem 1 (Constrained optimization I) (Sundaram, page 168)

Solve the following maximization problem:

$$\text{maximize } \log(x) + \log(y)$$
$$\text{subject to } x^2 + y^2 = 1$$
$$x \geq 0, y \geq 0$$

Problem 2 (Constrained optimization II) (Sundaram, page 168)

A firm produces two outputs y and z using a single input x. The set of attainable output levels $H(x)$ from an input use of x is given by

$$H(x) = \{(y, z) \in \mathbb{R}^2 | y^2 + z^2 \leq x\}$$

The firm has available to it a maximum of one unit of the input x. Letting p_y and p_z denote the prices of the two outputs, determine the firm's optimal output mix.

Problem 3 (Constrained optimization III) (Sundaram, page 169)

A consumer has income $I > 0$ and faces a price vector $p \in \mathbb{R}^3_{++}$ for the three commodities she consumes. All commodities must be consumed in nonnegative amounts. Moreover, she must consume at least two unites of commodity 2, and cannot consume more than one unit of commodity 1. Assuming $I = 4$ and $p = (1, 1, 1)$, calculate the optimal consumption bundle if the utility function is given by $u(x_1, x_2, x_3) = x_1 x_2 x_3$. What if $I = 6$ and $p = (1, 2, 3)$?

Problem 4 (Constrained optimization IV) (Sundaram, page 169)

Let $T \geq 1$ be some finite integer. Solve the following maximization problem:

$$\text{maximize } \sum_{t=1}^{T} \left(\frac{1}{2}\right)^t \sqrt{x_t}$$
$$\text{subject to } \sum_{t=1}^{T} x_t \leq 1$$
$$x_t \geq 0, t = 1, 2, ..., T$$

Problem 5 (Constrained optimization V)

Consider the following maximization problem:

$$\text{maximize } \alpha \log(x_1) + (1 - \alpha) \log(x_2)$$
$$\text{subject to } p_1 x_1 + p_2 x_2 \leq I, x \geq 0, y \geq 0 \tag{1}$$

p_1, p_2, and I are unknown parameters, and are all strictly positive.

a. Solve the problem by applying the Kuhn-Tucker theorem; be sure to include all three inequality constraints.

b. Solve the problem again by applying the theorem of Lagrange, according to the following outline: first, argue that the first constraint, $p_1x_1+p_2x_2 \leq I$, must hold with equality at any maximum. Second, apply the theorem of Lagrange to maximize $\alpha\log(x_1)+(1-\alpha)\log(x_2)$ over $\mathbb{R}^2_{++}\bigcap\{(x_1,x_2)\in\mathbb{R}^2 : p_1x_1+p_2x_2 = I\}$. Third, check the value of the objective function at the "endpoints," $(\frac{I}{p_x},0)$ and $(0,\frac{I}{p_y})$. This should give you sufficient justification to claim that you have solved the problem. Argue that your answer is the same as in part a., where you used the Kuhn-Tucker theorem.