Homework 9

due 11/17/08

Problem 1 (Identifying concave functions) (Sundaram, page 198) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = ax^2 + by^2 + 2cxy + d$. For what values of a, b, c and d is f concave?

Problem 2 (Properties of concave functions I) (Sundaram pg 198) Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be a concave function satisfying $f(0) = 0$. Show that for all $k \geq 1$ we have $kf(x) \geq f(kx)$. What happens if $k \in [0, 1)$?

Problem 3 (Identifying concave and convex functions) (Sundaram, page 198)

Show that the affine function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) = a \cdot x + b$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ is both convex and concave on $\mathbb{R}^n$. Conversely, show that if $f : \mathbb{R}^n \to \mathbb{R}$ is both concave and convex, then it is an affine function.

Problem 4 (Properties of concave functions II) (Sundaram pg 198) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be concave functions.

a. Give an example to show that their composition $f \circ g$ is not necessarily concave.

b. Is the product $f * g$ concave? Prove or provide a counter example.

Problem 5 (Constrained maximization) (Sundaram, pg 200) Let T be any positive integer. Consider the following problem:

$$\begin{aligned} \max \quad & \sum_{t=1}^{T} u(c_t) \\ \text{subject to} \quad & c_1 + x_1 \leq x \\ & c_t + x_t \leq f(x_{t-1}), t = 1, 1, ..., T \\ & c_t, x_t \geq 0, t = 1, 2, ..., T \end{aligned}$$

where $x \in \mathbb{R}_+$, and u and f are nondecreasing continuous functions from $\mathbb{R}_+$ into itself. Derive the Kuhn-Tucker conditions for this problem, and explain under what circumstances these conditions are sufficient.