Homework #3

answers

Problem 1 Let T be some finite integer. Solve the following maximization problem:

$$\max_{\{x_t\}} \sum_{t=1}^{T} \sqrt{x_t} \quad \text{subject to} \quad \sum_{t=1}^{T} x_t \leq 1,$$
$$x_t \geq 0, \ t = 1, 2, ..., T$$

Note that this is exactly a CES utility maximization problem with all prices and income set to 1. Therefore, we have:

$$x_1 = x_2 = ... = x_T = \frac{1}{T}$$

Problem 2 A consumer has income $I > 0$ and faces prices (p_1, p_2, p_3) for the three goods she consumes. Her utility function is $u(x_1, x_2, x_3) = x_1 x_2 x_3$. All goods must be consumed in nonnegative amounts. Furthermore, she must consume at least 2 units of good 2, and cannot consume more than 1 unit of good 1.

a. Assuming $I = 4$ and $(p_1, p_2, p_3) = (1, 1, 1)$, calculate her optimal consumption bundle.

If the consumer can purchase any bundle she wants, she would choose $x_1 = x_2 = x_3 = \frac{4}{3}$. As this violates the constraint that $x_1 \leq 1$, apply the Kuhn-Tucker theorem, treating $x_1 + x_2 + x_3 = 4$ as an equality constraint, and $x_1 \leq 1$ and $x_2 \geq 2$ as an inequality constraint. Ignore nonnegativity constraints $x_i \geq 0$, as they are clearly not going to bind. The necessary conditions for a max are given by

$$\begin{bmatrix} x_2x_3 \\ x_1x_3 \\ x_1x_2 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$x_1 + x_2 + x_3 = 4, \quad \lambda_1 \geq 0$$
$$\lambda_2(2 - x_2) = 0, \quad \lambda_1 \geq 0, \quad x_2 \geq 2$$
$$\lambda_3(x_1 - 1) = 0, \quad \lambda_1 \geq 0, \quad \lambda_3 \leq 1$$

Suppose $x_2 > 2$. Then $\lambda_2 = 0$, which implies $x_3 = x_2$, which is impossible if $x_2 > 2$. Conclude $x_2 = 2$ in any solution to the above equations. Suppose $x_1 < 1$. Then $\lambda_3 = 0$, which implies $x_1 = x_3$, which is inconsistent with $x_1 < 1$, $x_2 = 2$ and $x_1 + x_2 + x_3 = 4$. Conclude that $x_1 = 1$ in any solution. Evidently, then, the solution to the above system is $(x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3) = (1, 2, 1, 2, 1, 0)$. As the constraint qualification is satisfied and as the maximization problem clearly has some solution, the consumer maximizes her utility at $(x_1, x_2, x_3) = (1, 2, 1)$.

b. Now assume $I = 6$ and $(p_1, p_2, p_3) = (1, 2, 3,)$. What is her optimal consumption bundle?

A similar setup to part a tells us that the constraints $x_1 \leq 1$ and $x_2 \geq 2$ will bind with equality. The utility-maximizing bundle is at $(x_1, x_2, x_3) = (1, 2, \frac{1}{3})$.

Problem 3 Your ship is overdue in port and the beer is running out. The remaining supplies are divided up and you get 22.5 liters. The ship will not reach port before tomorrow morning, and there is a 60% chance that it will arrive then. You can't take beer with you when you leave the ship, so you could drink it all today, to make sure it isn't wasted. On the other hand, there is a 40% chance that you will still be afloat all day tomorrow, and a 10% chance that you will be afloat the day after that. You could save some beer in case you need it for the second day, or the third. It is certain you will reach port before the fourth day.

You are an expected utility maximizer, and your utility is $6000B - 250B^2$, where B is liters of daily beer consumption. How much beer should you drink today? How much should you save for tomorrow? For the day after tomorrow? (hint: your answers should be round numbers)

Setup the problem as $\max_{B_1,B_2,B_3} u_1 + .4u_2 + .1u_3$ subject to the constraint $B_1 + B_2 + B_3 = 22.5$. Solve using standard methods to get $B_1 = 11$, $B_2 = 9.5$, and $B_3 = 2$.

Problem 4 Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = ax^2 + by^2 + 2cxy + d$. For what values of a, b, c, and d is f a concave function?

As f is a polynomial, it is a C^2 function, and so the second derivative test for concavity is necessary and sufficient. Specifically, we need $D^2 f$ to be a n.s.d. matrix at all $(x, y) \in \mathbb{R}^2$, where

$$D^2 f(x, y) = 2 \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

This is a n.s.d. matrix so long as $a \leq 0$, $b \leq 0$, and $ab - c^2 \geq 0$.

Problem 5 A firm produces an output y using two inputs x_1 and x_2 as $y = \sqrt{x_1 x_2}$. Union rules obligate the firm to use at least one unit of x_1 in its production process. The input prices of x_1 and x_2 are w_1 and w_2, respectively. Assume that the firm wishes to minimize the cost of producing $\overline{y}$ unites of output.

a. Set up the firm's cost-minimization problem. Is the feasible set closed? Compact? Convex?

The firm's cost-minimization problem is given by

$$\min_{x_1,x_2} w_1 x_1 + w_2 x_2 \quad \text{subject to} \quad \sqrt{x_1 x_2} \geq \overline{y} \qquad (1)$$
$$x_1 \geq 1$$

The constraint set is closed and convex, but not bounded. However, any $(\tilde{x_1}, \tilde{x_2})$ pair satisfying both constraints costs at least as much as the cheapest input combination which yields $\overline{y}$ output, and so the constraint set can be further restricted to $\{(x_1, x_2) : \sqrt{x_1 x_2} \geq \overline{y}, x_1 \geq 1, w_1 x_1 + w_2 x_2 \leq w_1 \tilde{x_1} + w_2 \tilde{x_2}\}$, which is closed, bounded, and convex (this is identical to an argument seen in class, and may be best understood by drawing a picture).

b. Describe the Kuhn-Tucker first-order conditions. Are they sufficient for a solution? Why or why not?

Consider the minimization problem (**??**). Apply Sundaram theorem 7.16. To make the problem fit the theorem, we'll maximize the function $-w_1 x_1 - w_2 x_2$. The objective function is concave (by virtue of being linear). The constraint function $\sqrt{x_1 x_2} - \overline{y}$ is also concave (can check with second derivative condition; already proven as an example in class). Therefore, the K-T conditions are sufficient for a minimum, and the constraint qualification is irrelevant (clearly, there are many points such that the constraint holds strictly).

The Kuhn-Tucker conditions are given by:

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \lambda_1 \begin{bmatrix} \frac{1}{2}\sqrt{\frac{x_2}{x_1}} \\ \frac{1}{2}\sqrt{\frac{x_1}{x_2}} \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_1(\sqrt{x_1 x_2} - \overline{y}) = 0, \quad \sqrt{x_1 x_2} \geq \overline{y}, \quad \lambda_1 \geq 0$$

$$\lambda_2(x_1 - 1) = 0, \quad x_1 \geq 1, \quad \lambda_2 \geq 0 \tag{2}$$

Clearly, the first constraint binds with equality, as otherwise $\lambda_1 = 0$, which is inconsistent with the gradient condition (had the problem not specifically asked for K-T conditions, you would have been better off reducing it to an equality-constrained problem first).

Consider two cases. One, $x_1 > 1$. Then, $\lambda_2 = 0$, and the solution is clearly given by

$$x_1 = \overline{y}\sqrt{\frac{w_2}{w_1}}, \quad x_2 = \overline{y}\sqrt{\frac{w_1}{w_2}} \tag{3}$$

with cost function $c(w, y) = 2y\sqrt{w_1 w_2}$.

Now consider the case where $x_1 = 1$. Then $x_2 = \overline{y}^2$, and $c(w, y) = w_1 + w_2 y^2$.

The first case is relevant if

$$x_1 = \overline{y}\sqrt{\frac{w_2}{w_1}} > 1$$

and so the complete cost function is

$$c(w, y) = \begin{cases} 2y\sqrt{w_1 w_2} & \text{if } y\sqrt{\frac{w_2}{w_1}} \geq 1 \\ w_1 + w_2 y^2 & \text{if } y\sqrt{\frac{w_2}{w_1}} < 1 \end{cases} \tag{4}$$