Homework 5

answers

Problem 1 (hyperbolic discounting) Economists say that "hyperbolic discounting" describes the behavior of someone saying 'today I'll spend, and tomorrow I'll save' or 'I'll eat a big meal today, and start dieting tomorrow', day after day. Such attitudes are not time consistent, because when tomorrow rolls around, the person says the same thing. This type of model has been used to explain the behavior of smokers and the low US savings rate (see the papers of David Laibson).

Consider the following model of hyperbolic discounting, where $\gamma \in (0,1)$ and $\beta \in (0,1)$ are both discount factors.

$$\max_{\{c_t, w_{t+1}\}_{t=0}^{\infty}} log(c_0) + \gamma \left[\beta * log(c_1) + \beta^2 * log(c_2) + \beta^3 * log(c_3) + ...\right]$$

$$\text{subject to } c_t + w_{t+1} = A w_t^{\alpha} \;\; \text{for} \;\; t = 0, 1, 2, ...$$

$$w_0 \text{ given}$$

note that this is identical to the model solved in the 10/18 class except for the unusual discounting.

a. Go as far as you can in explaining why the above model does or does not fit the 'today I'll spend, but tomorrow I'll save' story discussed above.

b. Write down a value function $V[w]$ corresponding to the above problem (Hint: look at the example we did in the 10/18 class and note that the period t maximization problem is not the same as the period $t+1$ problem from the point of view of period t, so you need two different value functions).

Let $Z[w] = \max_{w'} \log(Aw^{\alpha} - w') + \beta Z[w']$, i.e. Z is the value function associated with the 'standard' problem solved in class. Then,

$$V[w] = \max_{w'} \log(Aw^{\alpha} - w') + \gamma\beta Z[w]$$

c. Derive a closed form solution for $V[w]$, using the guess-and-verify approach (again, look carefully at the standard model studied in the 10/18 class). In class, we showed, using guess and verify, that $V[w] = a + b\log(w)$, where $b = \frac{\alpha}{1-\alpha\beta}$, and a is some complicated constant. Thus,

$$V[w] = \max_{w'} \log(Aw^{\alpha} - w') + \gamma\beta a + \frac{\gamma\alpha\beta}{1-\alpha\beta}\log(w')$$

which has FOC

$$w' = \frac{\gamma\alpha\beta A w^{\alpha}}{1 - \alpha\beta(1-\gamma)}$$

and so a closed-form solution for V is

$$V[w] = \log\left(\frac{Aw^{\alpha}(1-\alpha\beta)}{1-\alpha\beta(1-\gamma)}\right) + \gamma\beta a + \frac{\gamma\alpha\beta}{1-\alpha\beta}\log\left(\frac{\gamma\alpha\beta A w^{\alpha}}{1-\alpha\beta(1-\gamma)}\right)$$

d. Define the *savings rate* to be $\frac{w'}{Aw^{\alpha}}$ to be the fraction of all output saved for the subsequent period. Does this savings rate change over time? Show that the actual savings rate in period t is different from that which was planned from the perspective of period $t-1$. At any time t, an agent practicing hyperbolic discounting plans on saving fraction $\alpha\beta$ in periods $t+1, t+2, ...$, but upon reaching any of those periods actually saves fraction $\frac{\gamma\alpha\beta}{1-\alpha(1-\gamma)} < \alpha\beta$.

Problem 2 (Guess and verify) A vintner[1] has one unit of labor to use each day. He can allocate that labor between the making of bread and the pressing of grapes for grape juice. The bread he makes today he can consume today. The grape juice he makes today will become tomorrow's wine (he doesn't care for grape juice). The production technology is linear: it produces one unit of bread per unit of labor allocated to baking, and one unit of juice per unit of labor allocated to grape pressing, and one unit of wine per unit of grape juice left to ferment. The transformation of juice into wine requires no labor, only time. The vintner allocates his labor so as to maximize the utility of his own consumption. His utility function has the form:

$$\sum_{t=0}^{\infty} \beta^t \sqrt{b_t w_t}$$

where b_t and w_t are the bread and wine consumption, respectively, in period t. The initial wine consumption w_0 is given. The discount factor is $\beta \in (0, 1)$.

a. Write down the value function associated with this problem. What are its control variable(s)? State variable(s)?

The sole state variable is w, the amount of wine he has at the beginning of a period. The control variables are b and w', how much bread he makes for today and how much wine he makes for tomorrow[2]. Noting that $b = 1 - w'$, the Bellman equation is then given by $V[w] = \max_{w'} \sqrt{w(1-w')} + \beta V[w']$.

b. Guess that the value function has the form $V(w) = \alpha\sqrt{\gamma + w}$, where α and γ are unknown parameters, and go as far as you can in verifying that V has this form. Solve for parameters α and γ.

Under the guess,

$$V[w] = \max_{w'} \sqrt{w(1-w')} + \alpha\beta\sqrt{w' + \gamma}$$

This maximization problem has FOC

$$w' = \frac{(\alpha\beta)^2 w - w^2\gamma}{w^2 + (\alpha\beta)^2 w}$$

and so, if the guess is correct, we have that

$$\begin{aligned} V[w] &= w\sqrt{\frac{1+\gamma}{w + (\alpha\beta)^2}} + (\alpha\beta)^2\sqrt{\frac{1+\gamma}{w + (\alpha\beta)^2}} \\ &= \sqrt{1+\gamma}\sqrt{w + (\alpha\beta)^2} \end{aligned}$$

This verifies the guess, with $\alpha = \sqrt{\frac{1}{1-\beta^2}}$ and $\gamma = \frac{\beta^2}{1-\beta^2}$.

c. State the optimal choice of next period's wine, w', as a function of current wine w.

Substituting our solutions for α and γ into the FOC for w', we get $w'(w) = \frac{\beta^2(1-w)}{w(1-\beta^2)+\beta^2}$.

A useful further exercise would be to argue that a value function exists for this problem by proving the Bellman equation defines a contraction, using Blackwell's sufficiency conditions.

[1] vintner=one who makes wine

[2] Technically, the amount of labor devoted to bread production, the amount to grape juice production, and total grape juice produced could all be though of as control variables, but as they are so immediately redundant with b and w', it is easiest to not even mention them.