Midterm #2

date: k periods before Apocalypse

Instructions: Each of three questions is worth 25 points. You have 75 minutes to complete the exam. Make sure to fully support your answers. Note that there are questions printed on both sides of this sheet.

Problem 1 Consider the Bellman equation below:

$$V[w] = \max_{c,w'} \ln(c) + \beta V[w']$$

$$\text{subject to } c + w' = Aw^\alpha \text{ and } w_0 > 0 \text{ given}$$

Recall (from class) that the optimal policy rule for this problem is given by $w'(w) = \alpha\beta A w^\alpha$. Suppose $A = 1.2$, $\alpha = 1$, and $\beta = .9$.

a. Show that optimal consumption in period t converges to ∞ as $t \to \infty$, for any w_0.

$c(w) = Aw^\alpha - w'(w) = (1-\alpha\beta)Aw^\alpha$. Under the given parameterization, $c(w) = .12w$, and $w'(w) = 1.08w$, so $c_t = .12 * 1.08^t * w_0$, which clearly goes to ∞ in t.

b. A friend tells you that you must have made a mistake. His argument is that if $\lim_{t\to\infty} c_t = \infty$, then clearly $\lim_{t\to\infty} u(c_t) = \infty$, but since total utility is infinite, the problem is degenerate, as small deviations from the optimal policy rule will also yield infinite utility. Your friend is wrong. Explain why.

He is right that $\lim_{t\to\infty} u(c_t) = \infty$. $u(c_t) = \ln(1.2 * 1.08^t * w_0) = t * \ln(1.08) + \ln(1.2w_0)$, which clearly goes to infinity, but at a linear rate. However, *discounted* utility, $\beta^t u(c_t)$ in fact goes to zero, as β^t approaches 0 geometrically, i.e. faster than $u(c_t)$ approaches ∞. Therefore, $\lim_{t\to\infty} \beta^t u(c_t) = 0$. (Actually, some more work is needed to show that $\sum_{t=0}^{\infty} \beta^t u(c_t)$ is finite —e.g. a root or ratio test— but what we have is sufficient to poke a hole in your friend's argument.)

Problem 2 A single-good monopolist maximizes the present value of profits. The inverse demand curve for the monopolist's product is $p = 50 - \frac{1}{20}q$. The monopolist's marginal cost per unit of output is $20 - 20 * \frac{e}{e+10}$, where e is the monopolist's production experience, and follows $e_{t+1} = e_t + q_t$. The monopolist has no fixed cost. The monopolist discounts t-period-ahead profits by $\left(\frac{1}{R}\right)^t$, where R is the interest rate. His time horizon is infinite.

a. Write down the monopolist's Bellman equation. Be very clear about what his state variable(s) and control variable(s) are.

The Bellman equation is below:

$$V[e] = \max_{e',q} q\left(50 - \frac{1}{20}q\right) - q\left(20 - \frac{20e}{e+10}\right) + \frac{1}{R}V[e']$$

$$\text{subject to } e' = q + e \tag{1}$$

You can also substitute the constraint into the Bellman equation to eliminate one of the control variables.

b. Do you think the monopolist's value function V is increasing or decreasing in its argument(s)? Why?

I think it is increasing. As e increases, the monopolist's marginal cost of production decreases, which can only increase the monopolist's profits.

c. Make a guess for V that satisfies the property you identified in b., and verify whether or not it is the correct guess.

I guess $V[e] = e$. To verify, plus the guess into next period's value function in the Bellman equation:

$$V[e] = \max_{e'} (e' - e)\left(50 - \frac{e' - e}{20}\right) - (e' - e)\left(20 - \frac{2(e' - e)}{e + 10}\right) + \frac{1}{R}e' \tag{2}$$

This has first-order condition:

$$(50 - \frac{e' - e}{20}) - \frac{e'}{2} - 20 + \frac{2(e' - e)}{e + 10} + (e' - e)\frac{2}{e + 10} + \frac{1}{R} = 0$$

$$\Rightarrow e'(e) = \frac{\frac{e}{20} + \frac{4e}{e+10} - 30 - \frac{1}{R}}{\frac{4}{e+10} - \frac{1}{2} - \frac{1}{20}}$$

Plugging the FOC into (**??**) will not result in the equation $V[e] = e$, so the guess is disconfirmed.

d. Suppose $e = 0$. Compare this monopolist's choice of output q_0 to that of a *myopic* monopolist, who is identical in every way except that $e' = 0$ (he does not gain production experience no matter how much he produces). Who produces more in period 0?

This monopolist will produce more in period 0, as he has the additional marginal benefit of producing of lowering his costs in future periods. One way to look at this is that the FOC for this monopolist's choice of q_0 is $MR + \frac{1}{R}V'(e) = MC$, while a myopic monopolist sets $MR = MC$. Since $V'(e)$ is positive, and since marginal revenue is decreasing in q, the monopolist in this problem will produce more than a myopic monopolist.

Problem 3 A consumer solves the following utility-maximization problem:

$$u(X,Y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 - 5x - 2y \quad \text{subject to } 5x + 2y \le 58 \tag{3}$$
$$x \ge 0$$
$$y \ge 0$$

It is clear a solution exists, as u is continuous, and the constraint set is compact. You may take as given that there are zero issues with the Kuhn-Tucker condition applying here (i.e. the rank condition is satisfied), so the max appears as a point satisfying the Kuhn-Tucker conditions.

a. Fact: there are two points (x, y) satisfying the Kuhn-Tucker conditions for solving (**??**). Find them, using your choice of the following two methods:

1. Write down the Kuhn-Tucker conditions and find two points that satisfy all of them jointly.[1]

2. Exploit this problem's structure to draw a picture, with the two points labeled. A brief explanation should accompany the picture.

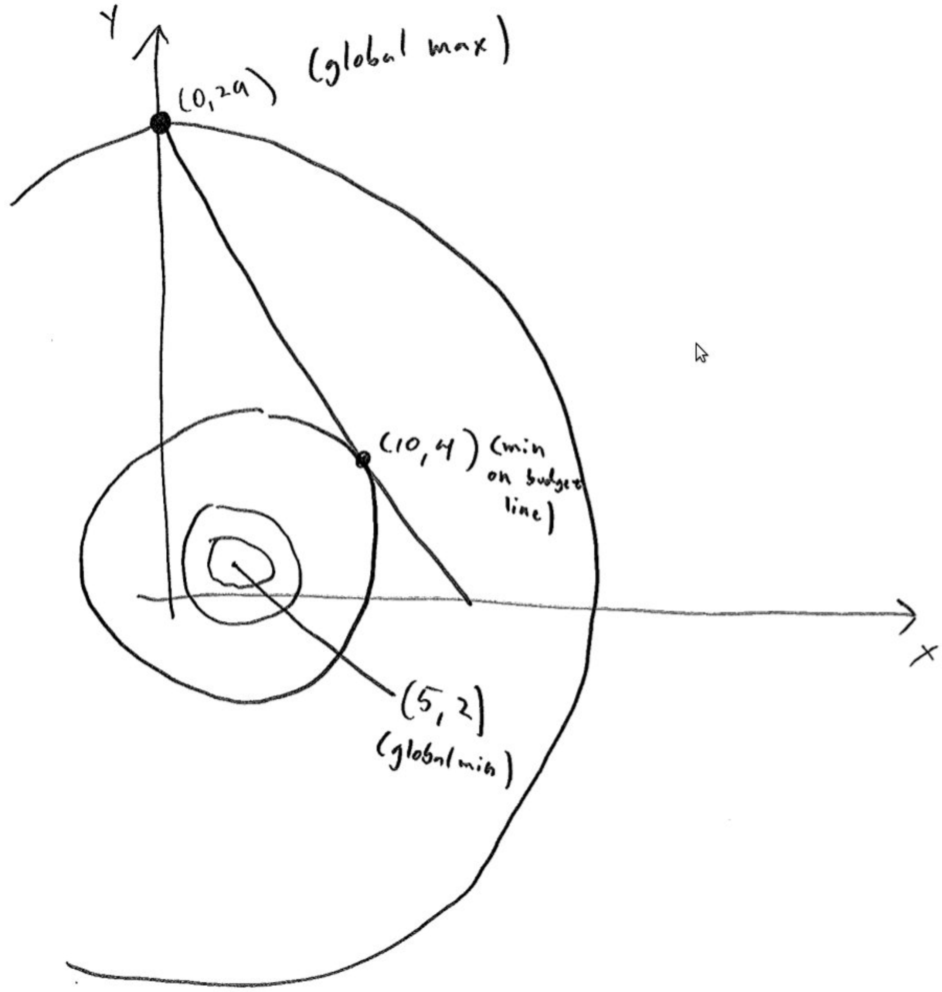
First, I intended to put a negative sign in front of the utility function, so that utility would be maximized, instead of minimized, at $x = 5$, $y = 2$. As I neglected to do this, the problem changes somewhat from what I intended. Fortunately, the problem is still coherent *except* that there are more than two points satisfying the KT conditions. Answers that are reasonable in light of this constraint will receive credit.

I choose method 1. See figure on next page. I will draw a picture of indifference curves over the budget set. Clearly, utility is minimized at $x = 5$, $y = 2$, which is inside the constraint set (to see this most clearly, note that this utility function is a monotonic transformation away from $u = (x-5)^2 + (y-2)^2$, which is always positive, but zero at $x = 5$, $y = 2$.) Therefore, $x = 5$, $y = 2$ will satisfy the Kuhn-Tucker conditions. Second, utility is increasing in the distance from point $x = 5$, $y = 2$. Indifference curves are therefore circles centered at $x = 5$, $y = 2$. There is a point of tangency between the budget line and one of these indifference curves at the point of minimum distance from $x = 2$, $y = 5$ to the budget line. This is the point where $\frac{MU_X}{MU_Y} = \frac{P_X}{P_Y}$, or $x = 10$, $y = 4$. $x = 10$ $y = 4$ is the utility-minimizing bundle on the budget line.

b. Which of your two points is the global max? Can you tell if the other is a local max or a local min?

It looks like there are 5 candidate points: $(5, 2)$, $(10, 4)$, $(0, 0)$, $(0, 29)$, and $(\frac{58}{5}, 0)$. Clearly, $(0, 29)$ gives the highest utility, and so is the global max. $(5, 2)$ is clearly a global min. $(10, 4)$ is a local minimizer of utility on the budget constraint. $(\frac{58}{5}, 0)$ is a local maximizer of utility. $(0, 0)$ is a local maximizer. These are all best seen via the picture.

[1] Formally, find two (x, y) pairs for which there exists a $(\lambda_1, \lambda_2, \lambda_3)$ vector such that $(x, y, \lambda_1, \lambda_2, \lambda_3)$ satisfy all Kuhn-Tucker conditions.


Y
(0,2a) (global max)
(10,4) (min on budget line)
(5,2) (global min)
X