# Homework 1
### due 1/26/17

**Problem 1** Kirt and Lila are engaged in a joint project. If person $i \in \{K, L\}$ invests effort $x_i \in [0, 1]$ in the project, at cost $c(x_i)$, the outcome of the project is worth $f(x_K, x_L)$. The worth of the project is split equally by Kirt and Lila, regardless of their effort levels, so that each gets a payoff of $\frac{1}{2}f(x_K, x_L) - c(x_i)$. Suppose effort levels are chosen simultaneously.

**a.** Suppose $f(x_K, x_L) = 3x_K x_L$ and that $c(x_i) = x_i^2$. Find the Nash equilibrium effort levels of this simultaneous move game.

K solves the following, taking $x_L$ as given: $\max_{x_K} \frac{3}{2}x_K x_L - x_K^2$, which has a maximum located at $x_K^* = \frac{3}{4}x_L$. Similarly, L's payoff is maximized at $x_L^* = \frac{3}{4}x_K^*$. The only values that satisfy both of these equations are $(x_K^*, x_L^*) = (0, 0)$. Therefore, the unique Nash equilibrium of this game is located at $(0, 0)$.

**b.** Is there a pair of effort levels that yield higher payoffs for both players than do the Nash equilibrium effort levels in part a.?

In the Nash equilibrium described in part a, both players get a payoff of 0. Now consider the alternative arrangement $x_K = x_L = 1$. Here, each player gets a payoff of $\frac{1}{2}$, and is therefore better off than in the Nash equilibrium.

**Problem 2** Consider the normal form game below:

Avon

|  |  | $I$ | $N$ |
|---|---|---|---|
|  | $I$ | $r, r$ | $r - 1, 0$ |
| Joe | $N$ | $0, r - 1$ | $0, 0$ |

In this game, strategy $I$ represents investing, and strategy $N$ represents not investing. Investing yields a payoff of $r$ or $r - 1$, according to whether the player's opponent invests or not. Not investing yields a certain payoff of 0.

Describe the set of Nash equilibria (pure and mixed) of the game for each $r \in [-2, 3]$.

First, if $r \in (1, 3]$, strategy $I$ is dominant for both players, and so $r, r$ is the unique Nash equibrium. Likewise, if $r \in [-2, 0)$, $N$ is strictly dominant for each player, and so $N, N$ is the unique Nash equilibrium. If $r \in [0, 1]$, then both $(I, I)$, $(N, N)$, and $((1 - r)I + rN, (1 - r)I + rN)$ are Nash equilibria, with the last being a mixed equilibrium in which each player plays $I$ with probability $1 - r$.

**Problem 3** Gibbons, problem 1.13

This is similar to the Hawk-Dove game studied in class. There are two pure-strategy Nash equilibria, in which the workers apply to different jobs (1 applies to firm 1, 2 applies to firm 2; 1 applies to firm 2, 2 applies to firm 1). There is also a mixed strategy Nash equilibrium, in which each worker applies to firm 1 with probability $\frac{2w_1 - w_2}{w_1 + w_2}$ and to firm 2 with probability $\frac{2w_2 - w_1}{w_1 + w_2}$. It is simple to verify that each of these probabilities is between 0 and 1.

**Problem 4** This problem demonstrates a seeming peculiarity about mixed strategy Nash equilibria. Consider the following game between the Chicago Bears' offense and the Detroit Lions' defense. Payoffs are the number of yards advanced (positive yards for Chicago are negative yards for Detroit).

|          |      | Detroit |      |
|----------|------|---------|------|
|          |      | run defense | pass defense |
| Chicago  | run  | -2,2    | 5,-5 |
|          | pass | 15,-15  | 1,1  |

**a.** Find all pure strategy Nash equilibria, if any. Then find the mixed-strategy Nash equilibrium of the game.

There are no pure strategy Nash equilibria. There is a mixed-strategy Nash equilibrium where Chicago runs fraction $\frac{16}{23}$ of the time, and passes $\frac{7}{23}$ of the time, and Detroit plays a run defense fraction $\frac{4}{21}$ of the time, and a pass defense $\frac{17}{21}$ of the time.

**b.** Now suppose that the Bears improve their run game by bringing Mike Ditka[1] out of retirement:

|          |      | Detroit |      |
|----------|------|---------|------|
|          |      | run defense | pass defense |
| Chicago  | run  | -2,2    | 10,-10 |
|          | pass | 15,-15  | 1,1  |

Find the mixed-strategy Nash equilibrium of the new game.

Now Chicago runs fraction $\frac{4}{7}$ of the time, and passes $\frac{3}{7}$ of the time, while Detroit plays a run defense fraction $\frac{9}{26}$ of the time, and a pass defense fraction $\frac{17}{26}$ of the time.

**c.** When running the football becomes a more attractive option for the Bears, do they run more often, or pass more often? Can you explain why?

The Bears run less often when their running game improves. The reason is that after Mike Ditka signs with the Bears, the Lions will become relatively more inclined to play a run defense, lowering the Bear's yardage on run plays.

**Problem 5** This problem refers to the following game:

|     | A    | B    | C    |
|-----|------|------|------|
| A   | 4,4  | 0,5  | -1,0 |
| B   | 5,0  | 1,1  | 0,0  |
| C   | 0,-1 | 0,0  | 1,1  |

**a.** What are the pure-strategy Nash equilibria?

$(B, B)$, and $(C, C)$.

**b.** Is there a mixed strategy Nash equilibrium where both players mix A and B? If so, find the equilibrium. If not, explain why not.

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[1]While Ditka played tight end, the combination of his blocking and the downfield threat he poses as a receiver, even at 77, would help their running game immeasurably.

No. B strictly dominates A for player 1, yet in a mixed Nash equilibrium 1 would need to be indifferent between A and B, which is impossible.

**c.** Is there a mixed-strategy Nash equilibrium where both players mix B and C? If so, find the equilibrium. If not, explain why not.

There is a mixed strategy Nash equilibrium, in which both players play $(\frac{1}{2}B + \frac{1}{2}C)$.

**Problem 6** In the game below, which strategies survive iterated removal of strictly dominated strategies? What are the pure strategy Nash equilibria?

|   | L | C | R |
|---|---|---|---|
| T | 1,3 | 5,4 | 4,2 |
| M | 2,3 | 3,1 | 3,2 |
| B | 3,5 | 4,7 | 1,4 |

Only (T,C) survives iterated removal of strictly dominated strategies. This implies that (T,C) is the unique Nash equilibrium.

**Problem 7** 99 shepherds share a common field in which they graze their sheep. Each shepherd purchases as many sheep as he/she likes, at a cost of $c = \$300$/sheep. The value of one sheep is given by:

$$v(G) = 2000 - S$$

where $S$ is the total number of sheep which graze in the field (more sheep mean less grass/sheep, more sheep fights, etc). The common field is the only suitable location for grazing, and sheep die without grazing, so you may assume that all purchased sheep are brought to graze in the field.

**a.** In a symmetric Nash equilibrium, how many sheep does each shepherd purchase? How much profit is earned by each shepherd?

Each shepherd will purchase 17 sheep, and earn a profit of $289.

**b.** What is the socially optimal number of sheep? If the resulting total profit is split evenly amongst all shepherds, what is the profit for each shepherd?

In a social optimum, there are 850 total sheep, or 8.58/shepherd. If the total profit from 850 sheep is split among the 99 shepherds, each will earn $7,297.98.

**c.** Suppose a government imposes a tax on sheep of $T/head, but that the revenue collected from the tax is distributed evenly to each of the 99 shepherds, regardless of how many sheep the shepherd owns. Is such a tax always welfare-reducing? Why or why not?

Such a tax can internalize the externality created when a shepherd purchases an additional sheep (namely, that the sheep lower the value of the other 98 shepherds' sheep). Specifically, a tax of $T$ will change the equilibrium number of sheep purchased by each shepherd to $17 - \frac{T}{100}$. So long as $T \leq \$841.41$ (the tax which would produce the socially optimal number of sheep in Nash equilibrium), the tax is efficient. Even if $T > \$841.41$, the tax may be preferable to no tax.