Homework 5

due 4/6/2017

Problem 1 Suppose that *low-ability* workers have productivity of D, while *high-ability* workers have productivity of A, where $A > D$. Firms cannot tell low-ability workers from high-ability workers *ex ante*, but can observe a worker's education level e. Firms know that half of all workers are low-ability, and half are high-ability.

Any worker can acquire as much education as she wishes, but getting e units of education costs a low-ability worker $B*e$, where $B > 1$, and costs a high-ability worker e. Assume the labor market is competitive, so that a worker earns her expected productivity.

a. Suppose $A = 15$, $B = 4$, and $D = 1$. Does there exist a pooling equilibrium in which both high- and low-ability workers get 1 unit of education? If so, draw or describe a wage function that supports this equilibrium outcome, and determine whether or not the equilibrium satisfies the intuitive criterion. If not, explain why not.

If all workers pool on $e = 1$, they would earn a wage of 8, and so high types would receive utility of 7, and low types utility of 4. The wage function that is most favorable to this being a pooling equilibrium is $w(1) = 8$ and $w(e) = 1$ for all $e \neq 1$. Under this wage function, the most profitable deviation for either type of worker is to $e = 0$. Such a deviation would give both high and low types utility of 1, which is less than their equilibrium utility. Conclude that there is a pooling equilibrium at $e = 1$, supported by the wage function described here. No pooling equilibrium satisfies the intuitive criterion. The listed wage function fails the intuitive criterion, because (for example) the wage function implies firms believe that a worker with 3 units of education is certainly a low type, but the maximal utility a low type can achieve with 3 units of education is $15 - 3 * 4 = 3$, which is less than their equilibrium payoff of 4.

b. Suppose $A = 15$, $B = 4$, and $D = 1$. Does there exist a pooling equilibrium in which both high- and low-ability workers get 3 units of education? If so, draw or describe a wage function that supports this equilibrium outcome, and determine whether or not the equilibrium satisfies the intuitive criterion. If not, explain why not.

In this case, low types receive utility of -4, and so would prefer to deviate to $e = 0$, where they would get at least 1 unit of utility. No matter the wage function, there is no equilibrium in which workers pool at $e = 3$.

c. Suppose $A = 15$, $B = 4$, and $D = 1$. Solve for a separating equilibrium which *does not* satisfy the intuitive criterion. Draw or describe a wage function that supports this outcome in an equilibrium. Clearly explain, using your graph and/or a verbal description, why this equilibrium fails the intuitive criterion.

From the answer to part d., in the unique equilibrium satisfying the intuitive criterion, high types choose $e = 3.5$.

Consider, then, the separating equilibrium in which $e_H = 4$, $e_L = 0$, $w(4) = 15$, and $w(e) = 1$ for all $e \neq 4$. Both types are optimizing and the firm has correct beliefs, so this is an equilibrium. However, the firm pays a wage of 1 in response to $e = 3.6$, and thus must believe that a worker acquiring 3.6 units of education is a low type with probability 1. However, any low type who acquires more than 3.5 units of education receives a utility of less than 1. Since 1 is the low type's equilibrium payoff, the firm's belief is disallowed by the intuitive criterion.

d. For general A, B, and D, solve for the unique equilibrium which *does* satisfy the intuitive criterion as a function of A, B and D. How does the level of education obtained by the high types vary in D in this equilibrium? What is the intuition?

In the intuitive criterion equilibrium, high types choose e_H units of education, where e_H leaves low types indifferent between $e_L = 0$ and $e_L = e_H$, or $D = A - B * e_H$, or $e_H = \frac{A-D}{B}$. As D increases, the equilibrium value of e_H decreases; this is because less education is needed to separate from low types, who are now paid a higher wage since they are more productive.

Problem 2 This problem asks you to consider an extension of the basic Spence model to one in which education is productive and the cost of education is convex.

Suppose that high types with education e have productivity $y(H, e) = 10 + 2e$, while low types have productivity $y(L, e) = 2 + e$. Firms cannot observe whether a worker is a high type or a low type, but know that half of all workers are of each type. A competitive labor market ensures each type of worker is paid her expected productivity. A high type can acquire e units of education at cost $c_H(e) = \frac{1}{10}e^2$, while education costs a low type $c_L(e) = \frac{1}{4}(e + 2)^2 - 1$.

a. Suppose $e_L = 0$ and $e_H = 12$. What wage function would support this outcome as a separating equilibrium? Draw a picture and/or describe using an equation.

Consider the following wage function:

$$wage(e) = \begin{cases} 34 & \text{if } e = 12 \\ 2 + e & \text{if } e \neq 12 \end{cases} \tag{1}$$

High types will maximize their utility by choosing $e = 12$ (giving utility of 19.6), while low types will maximize their utility by choosing $e = 0$ (giving utility of 2, versus -14 from deviating to $e = 12$).

b. Does the equilibrium you described in part a satisfy the intuitive criterion? Why or why not?

No. Consider education level $e = 10$. If $wage(e = 10) = 30$, high types would prefer $e = 10$ to $e = 12$, since their utility would increase to 20. Low types would not like to switch to $e = 10$ from $e = 0$ regardless of how high the wage is, since even if $wage(10) = 30$, low types utility from choosing $e = 10$ would be only -5.

c. Draw the set of all points which give the high type utility of 5. What is the slope of the indifference curve you drew, as a function of e? Determine the point of tangency between the high type's indifference curve and the function $y(H, e)$ (note that the high type may get more or less than 5 utility at the point of tangency). Do the same for the low type's indifference curve and the function $y(L, e)$.

The slope of the high type's indifference curve is $\frac{1}{5}e$, while the slope of her productivity function is 2. The point of tangency is then at $e = 10$. For the low types, the slope of the indifference curve is $\frac{1}{2}(e + 2)$, while the slope of his productivity function is 1, meaning to point of tangency is at $e = 0$.

d. Suppose that both types choose their education level so that their indifference curve is tangent to their productivity function. Describe, using a picture and/or an equation, a wage function that would support this outcome as a perfect Bayesian equilibrium.

Consider the following wage function:

$$wage(e) = \begin{cases} 30 & \text{if } e = 10 \\ 2 + e & \text{if } e \neq 10 \end{cases} \tag{2}$$

Low types strictly prefer $e = 0$ to all other education levels, and high types strictly prefer $e = 10$ to all other education levels (since they are by definition on their highest achievable indifference curve over all the points on their productivity function $10 + 2e$).

e. Does the equilibrium you described in part d satisfy the intuitive criterion? Why or why not?

Yes. Any education level other than $e = 10$ makes the high type worse off, so the intuitive criterion has no bite. (There are education levels that could potentially make only the low type better off, such as $e = 1$, but requiring $\mu(L|e = 1) = 1$ does not affect the equilibrium outcome; indeed, exactly this belief is embedded in the wage function described in part d.).

Problem 3 Workers choose their education levels and then apply for jobs. $\frac{1}{4}$ of workers are *high-ability*, while the remainder are *low-ability*. Firms cannot determine a given worker's ability level until after the worker is hired. Regardless of education level, high ability workers increase firm profits by \$100, while low ability workers increase firm profits by \$36. The cost of education is $c_L(e) = e^2$ for low-ability workers, while it is $c_H(e) = 2e$ for high ability workers. Assume that workers are paid their expected productivity.

For outcomes 1-4 below, determine if each outcome is an equilibrium of the game. If so, determine whether or not it satisfies the intuitive criterion, and prove your answer.

a. Outcome 1:

$$\begin{aligned} e_H &= 12 \\ e_L &= 2 \\ w(e) &= \begin{cases} \$100 & \text{if } e = 12 \\ \$36 & \text{if } e \neq 12 \end{cases} \end{aligned} \tag{3}$$

Not an equilibrium (low types prefer to switch to $e = 0$).

b. Outcome 2:

$$\begin{aligned} e_H &= 1 \\ e_L &= 1 \\ w(e) &= \begin{cases} \$52 & \text{if } e < 25 \\ \$100 & \text{if } e \geq 25 \end{cases} \end{aligned} \tag{4}$$

Not an equilibrium (low types prefer to switch to $e = 0$).

c. Outcome 3:

$$\begin{aligned} e_H &= 5 \\ e_L &= 5 \\ w(e) &= \begin{cases} \$36 & \text{if } e < 5 \\ \$52 & \text{if } e = 5 \\ \$(32 + 4e) & \text{if } e \in (5, 17) \\ \$100 & \text{if } e \geq 17 \end{cases} \end{aligned} \tag{5}$$

Not an equilibrium. Low types prefer to switch to $e = 0$.

d. Outcome 4. For this part, your answer should depend on the variables X and Y.

$$\begin{aligned} e_H &= X \\ e_L &= Y \\ w(e) &= \begin{cases} \$36 & \text{if } e < Y \\ \$100 & \text{if } e \geq Y \end{cases} \end{aligned} \tag{6}$$

Equilibrium if $X = Y$ and $Y \leq 32$. Satisfies intuitive criterion if $Y = 8$.

Problem 4 An incumbent firm is either a low-cost type (θ_L) or a high-cost type (θ_H), each with equal probability. In period 1, the incumbent is a monopolist and sets one of two prices, p_L or p_H, with its profits in period 1 given by the following table:

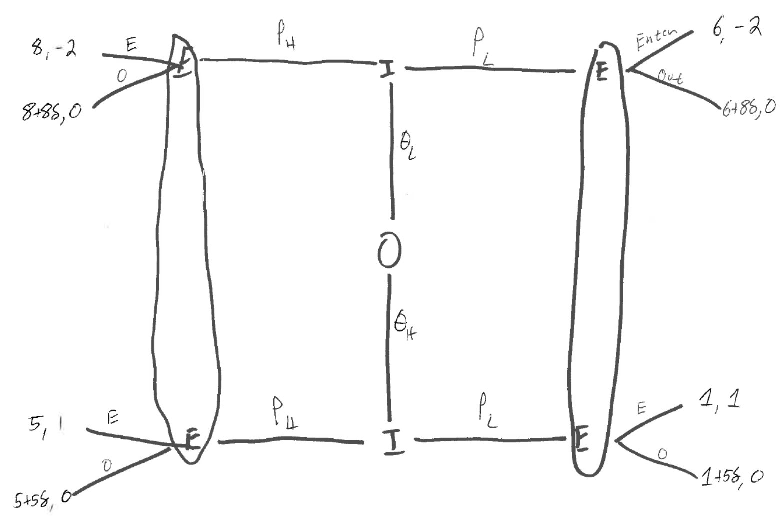
Type	Profit from p_L	Profit from p_H
θ_L	6	8
θ_H	1	5

After observing the period 1 price, a potential entrant (which does not know the incumbent's type) can choose to enter the market (E) or to stay out (O) in period 2. The payoffs of both players in period 2 are as follows:

Type	Entrant's choice	Incumbent's profit	Entrant's profit
θ_L	E	0	-2
θ_L	O	8	0
θ_H	E	0	1
θ_H	O	5	0

At the beginning of the game the incumbent discounts period 2 profits using discount factor $\delta \leq 1$.

a. Draw the extensive form game. Make sure to include all payoffs, as a function of δ.



Note that it is also acceptable to multiply the entrant's payoffs by δ. Since they all occur in period 2, however, doing so is unnecessary.

b. For the special case of $\delta = 1$, find a pooling perfect Bayesian equilibrium in which both incumbent types choose p_L in period 1. Does your equilibrium satisfy the intuitive criterion?

I plays E at his left information set and O at his right information set. Both types of I play P_L. Entrant believes he is at the bottom node of his left information set and is equally likely to be at the top and bottom nodes of his right information set.

c. Find the range of discount factors for which a separating equilibrium exists in which type θ_L chooses p_L and type θ_H chooses p_H in period 1.

$\delta \in [\frac{1}{4}, \frac{4}{5}]$.

Problem 5 Consider the game in Figure 1 below.

a. Draw the reduced normal form. Find all pure strategy Nash equilibria. There is a mixed Nash equilibrium in which 1 randomizes between A and B, and 2 randomizes between L and R. Find it.

One pure Nash equilibrium, at (C, M). The mixed Nash equilibrium is at $(\frac{1}{2}A + \frac{1}{2}B, \frac{1}{2}L + \frac{1}{2}R)$.

b. Find all of the game's perfect Bayesian equilibria (pure as well as mixed).

The one PBE is $(\frac{1}{2}A + \frac{1}{2}B, \frac{1}{2}L + \frac{1}{2}R)$, with 2 believing that each node is equally likely.

c. Explain in intuitive terms any differences between your answers to part a and part b.

The pure Nash equilibrium involves 2 playing a strictly dominated strategy. PBE rules this out.

Figure 1: Game for question 4