Midterm

2/23/2017

Instructions: You may use a calculator and scratch paper, but no other resources. In particular, you may not discuss the exam with anyone other than the instructor, and you may not access the Internet, your notes, or books during the exam.

If you don't know how to answer a question, go as far as you can. Sometimes substantial points can be awarded for the right setup, an intuitive explanation, or the right approach demonstrated on a simplified version of the problem. Similarly, if a problem requires multiple steps, it is important that you clearly describe your progression through those steps, even if you know the correct numerical answer. You have 120 minutes to complete the exam. Good luck!

Problem 1 (15 points) Refer to the Games 1 and 2 depicted in Figures 1 and 2.

a. Enter payoffs into the games below so that they have the following properties:

- The games are equivalent except for the addition of strategy C to game 2 (i.e. the first two rows of Game 1 are equivalent to the first two rows of Game 2).
- The only Nash equilibrium of Game 1 is (A, L).
- The only Nash equilibrium of Game 2 is (C, R).

Please copy the games onto your answer sheets to complete this question. There are many correct answers. One is given below.

Player 2

Player 1	L	R
A	3,3	0,0
B	0,0	-1,0

Figure 1: Game 1

Player 2

Player 1	L	R
A	3,3	0,0
B	0,0	-1,0
C	4,0	2,2

Figure 2: Game 2

b. Referring only to Game 1, enter payoffs such that the only Nash equilibrium is both players mixing with equal probability on each strategy.

Again, there are many possible correct answers. One example is given below.

Player 2

Player 1	L	R
A	3,0	0,3
B	0,3	3,0

Problem 2 (15 points) In 1897, Sears and Montgomery Ward operated two competing mail-order catalogs.

Suppose that demand, in thousands of units, for a certain type of coal-powered stove[1] sold in both catalogs, but not easily obtainable elsewhere, is given by $Q = 20 - \min\{P_S, P_M\}$. The firm with the lower catalog price supplies the entire market, while if prices are equal, the two firms each supply half of the market. Suppose that Sears and Montgomery Ward have identical marginal costs of \$10, which include all acquisition, storage, and shipping costs.

a. Sears and Montgomery Ward both release their catalogs on the same day, which means neither is aware of the other's price at the time it sets its own. What is the equilibrium price set by each catalog?

Sears and Montgomery Ward are Bertrand competitors; in the unique Nash equilibrium, each sets a price of \$10. To see that this is an equilibrium, note that each firm earns a profit of \$0. But if either firm were to lower its price, it would be losing money on each unit sold, and so would earn a negative profit. If either firm were to raise its price, its demand would fall to 0, and its profit would be no higher than at a price of \$10. Hence, each firm is playing a best response to the other's price.

Now, suppose there were an equilibrium in which firm A plays a price $X > \$10$. But then firm B's best response would be to price slightly under X, as this would give firm B the entire market at the highest possible price (unless X were above the monopoly price of \$15, in which case firm B would prefer a price of \$15). But then X cannot be a best response to firm B's price, and so a price of X cannot be part of a Nash equilibrium. Next, suppose there were an equilibrium in which firm A set a price of $X < \$10$. Here, B's best response would be to set any price $P_B > X$ and earn zero profit. But then firm A would lose money because of his negative margin, meaning that A is not playing a best response to firm B's price. Conclude that there is no equilibrium other than $P_A = P_B = \$10$.

b. Now suppose that in 1898, Montgomery Ward releases its calendar on January 1, but Sears releases its calendar on July 1, so that it can observe Montgomery Ward's posted price on this item before setting its own price. Would this change the equilibrium outcome from part a.?

Solve the game via backward induction. After observing any price $P_M > 10$, Sears' best response is $P_S = P_M - \epsilon$. After observing a price $P_M = \$10$, Sears' best response is $P_S = \$10$. After observing a price of $P_M < \$10$, Sears' best response is any $P_S > P_M$. Given these best responses, Montgomery Ward is indifferent over all prices $P_M \in [10, \infty)$, and so there are many possible equilibria. In each, Montgomery Ward earns 0 profit. In the equilibria in which $P_M > 10$, Sears earns a positive profit.

Modern internet retailers like Amazon scrape rivals' websites to continually track their prices; often, the scraping algorithm is empowered to automatically match rivals' prices. In game theoretic terms, Amazon is able to credibly commit to a strategy of matching its rivals' prices.

c. Suppose that amazon.com and walmart.com are duopoly sellers of a certain item,[2] with demand in thousands of units given by $Q = 20 - \min\{P_A, P_W\}$. Suppose that each has a marginal cost of \$10. Suppose that Wal-Mart believes that Amazon has committed to match its price exactly, whatever that price is. What price should Wal-Mart set?

In this case, Wal-Mart should set the monopoly price of $P = \$15$. Amazon will follow suit, and each will earn a profit of \$12.5.

[1] Specifically, the True Sunshine Range, with oven dimensions 15x16x10$\frac{1}{2}$ and a shipping weight of 225 pounds.
[2] Specifically, bandages that look like bacon strips.

d. Given your answer to part c., is amazon.com better off programming its algorithm to match walmart.com's price exactly, or to undercut it by 1 cent?

Amazon matching, rather than undercutting its competitors can facilitate collusion, in which each firm sets the monopoly price. Were Amazon to reprogram its algorithm to undercut rivals, and if the rivals had their own algorithms which undercut Amazon, the firms would force Bertrand competition (and marginal cost pricing) upon themselves.

Problem 3 (15 points) Consider a market with demand curve $P = 1 - Q$, served by Cournot oligopolists with marginal cost c who compete by simultaneously setting quantity.

a. Suppose that there are 2 firms, each with marginal cost $c = \frac{1}{4}$. Solve for the Nash equilibrium quantity, price, and profit for each firm.

In the Nash equilibrium, $q_1 = q_2 = \frac{1}{4}$. The price is $P = \frac{1}{2}$ and each firm earns a profit of $\frac{1}{16}$.

b. Now suppose that there are N firms, each with marginal cost $c = \frac{1}{4}$. Solve for the Nash equilibrium quantity, price, and profit for each firm.

In the Nash equilibrium, each firm produces $q = \frac{3}{4(N+1)}$. The market price is $\frac{4+N}{4(N+1)}$. Each firm earns a profit of $\frac{9}{16}\frac{1}{(N+1)^2}$.

Problem 4 (15 points) This problem refers to the following game:

	A	B	C
A	8,8	0,10	-2,0
B	10,0	2,2	0,0
C	0,-2	0,0	2,2

a. What are the pure strategy Nash equilibria?

A is strictly dominated by B for player 1. Removing this strategy, A is now strictly dominated by B for player 2, as well. In the remaining 2X2 game, it is direct that there are two pure strategy Nash equilibria, (B, B) and (C, C).

b. Is there a mixed strategy Nash equilibrium where both players mix A and B? If so, find the equilibrium. If not, explain why not.

No, as A is a strictly dominated strategy for player 1.

c. Is there a mixed strategy Nash equilibrium where both players mix B and C? If so, find the equilibrium. If not, explain why not.

Yes, there is a mixed equilibrium in which each player plays $\frac{1}{2}B + \frac{1}{2}C$.

Problem 5 (15 points) Consider the following game:

		Player 2 X	Y	Z
	A	3,3	-2,0	0,1
Player 1	B	5,-5	0,0	0,0
	C	0,0	$\frac{1}{2}$,0	1,1

a. Solve for all Nash equilibria, pure and mixed.

It is clear that A is strictly dominated by an equal mixture of B and C. But if A is removed, X is strictly dominated by Z for player 2. Once X is removed, C is a strictly dominant strategy for player 1. Hence, via iterated removal of strictly dominated strategies, the unique Nash equilibrium of this game is (C, Z).

b. Now suppose that this game is repeatedly infinitely often, and that both players have discount factor δ. For what values of δ do the following strategies comprise a subgame perfect equilibrium of the repeated game?

Phase I: Play (A, X) initially, and if (A, X) was played in the previous period. If either player deviates from (A, X) permanently move to Phase II in the following period.

Phase II: Play (C, Z).

As Phase II involves a Nash equilibrium of the stage game, and as player 2 has no incentive to deviate from (A, X), we need only determine the minimum value of δ for which player 1 does not wish to deviate from Phase I play. This is given by:

$$\frac{3}{1-\delta} \geq 5 + \frac{\delta}{1-\delta}$$
$$\iff \delta \geq \frac{1}{2} \tag{1}$$

c. Again suppose that the game is played repeatedly, but this time determine the minimum value of δ for which the following strategies comprise a subgame perfect equilibrium:

Phase I: Play (A, X) initially, and if (A, X) was played in the previous period. If either player deviates from (A, X) switch to Phase II in the following period.

Phase II: Play (B, Y) for 2 periods, and then switch to Phase I. If either player deviates from (B, Y), begin Phase II again.

Player 1 chooses not to deviate from Phase I if and only if:

$$\frac{3}{1-\delta} \geq 5 + \frac{3\delta^3}{1-\delta}$$
$$\iff \delta(1+\delta) \geq \frac{2}{3}$$
$$\iff \delta \geq \sqrt{\frac{11}{2}} - .5 = .457$$

Note that the last inequality can be approximated by trial and error if necessary. Now, only player 1 could possibly want to deviate from Phase II. Player 1's incentive constraint is given by:

$$\frac{3\delta^2}{1-\delta} \geq \frac{1}{2} + \frac{3\delta^3}{1-\delta}$$
$$\iff 3\delta^2 \geq \frac{1}{2}$$
$$\iff \delta \geq \frac{1}{\sqrt{6}} = .408$$

Conclude that the listed strategies form a SPE if and only if $\delta \geq .457$.

Problem 6 (15 points) Consider a 2-player bargaining game, in which the players are choosing how to split a surplus of \$1. Each player discounts payoffs one period in the future by $\delta = \frac{1}{2}$. The rules are as follows:

Period 0: Player 1 chooses a value of X between 0 and 1, at cost $c(X) = \frac{3}{8}X^2$.

Period 1: Player 2 makes an offer of $(y, 1-y)$, where y is between 0 and 1. Player 1 may either accept the offer, in which case the surplus is split accordingly and the game ends, or reject the offer, in which case the game moves on to period 2.

Period 2: Player 1 makes an offer to player 2. If the offer is accepted, the surplus is split accordingly and the game ends. If the offer is rejected, a mediator awards a payoff of X to player 1, and $1-X$ to player 2 (nb. this is the same X chosen by player 1 in period 0; interpret the cost player 1 incurred in that period as that of hiring an attorney to investigate the most favorable mediation venue).

a. Solve for the subgame perfect equilibrium of this game. Is player 1 or player 2 better off in this setup? Solve the game using backward induction. In period 2, player 2's outside option is $\delta * (1-X)$, so player 1's offer would be $(1-\delta+\delta X, \delta-\delta X)$, and it player 2 would accetp the offer. In period 1, player 2 would offer $(\delta-\delta^2+\delta^2 X, 1-\delta+\delta^2-\delta^2 X)$, and player 1 would accept the offer. In period 0, player 1 must equate the marginal cost of a higher X (which is $\frac{3}{4}X$ with the marginal benefit (which is δ^2). Given that $\delta = \frac{1}{2}$, then, we have that $X = \frac{1}{3}$. The subgame perfect equilibrium is thus player 1 chooses $X = \frac{1}{3}$ in period 0, and players 1 and 2 make offers as above in periods 1 and 2. Player 2's offer is accepted in period 1, and the game ends then. Player 1's payoff (from the perspective of period 1) is $\frac{1}{3}$, and player 2's payoff is $\frac{2}{3}$, so it is better to be player 2.

b. Intuitively, how would your answer change by an increase in δ? You do not need to work out the mathematical details.

For a constant $X > 0$, player 1's period 1 payoff of $\delta-\delta^2+\delta^2 X$ is increasing in δ. Further, in period 0, the marginal benefit of choosing a higher X increases as player 1 becomes more patient. Hence player 1 will receive a greater share of the surplus. Acceptable anwers could also recalculate the equilibrium outcome for one or two higher values of δ to gain intuition.