Homework 3

answers

Problem 1 Consider the game below:

	Cooperate	Defect
Cooperate	4,4	0,12
Defect	12,0	2,2

Suppose the game is played repeatedly, with both players sharing a discount factor $\delta \in (0, 1)$. For what range of values of δ do grim trigger strategies (cooperate so long as no one has defected, otherwise defect forever) comprise a subgame perfect equilibrium?

The punishment path involves repeating the stage game Nash equilibrium, so it is trivially a subgame perfect equilibrium. For cooperation to be sustainable in phase I of a subgame perfect equilibrium, we need:

$$\frac{4}{1-\delta} \geq 12 + \delta \frac{2}{1-\delta}$$
$$\iff \delta \geq \frac{4}{5}$$

So, the grim trigger strategies described above comprise a subgame perfect equilibrium if and only if $\delta \geq \frac{4}{5}$.

Problem 2 The inverse market demand curve for a good is given by $P = 1 - Q$. Two Cournot competitors compete to supply the market ($Q = q_1 + q_2$), each with zero marginal and fixed costs.

a. If the two firms were to collude to make joint profits as large as possible, what quantities would they produce, and what profit would they earn? You may assume that they each produce the same quantity.

A monopolist would charge price $p = \frac{1}{2}$ and sell quantity $q = \frac{1}{2}$, for profit of $\frac{1}{4}$. If the two firms here collude, they could each produce half of the monopoly quantity, or $q_1 = q_2 = \frac{1}{4}$, and each earn a profit of $\frac{1}{8}$.

b. Suppose one of the firms is considering unilaterally deviating from the collusive outcome in a. What is the maximal profit the firm can attain with such a deviation?

Suppose firm 2 is producing $q_2 = \frac{1}{4}$, as in part a. What quantity should should firm 1 produce to maximize its short-term profit? In this case, firm 1 solves the following maximization problem:

$$\max_{q_1} \left(1 - q_1 - \frac{1}{4}\right) q_1$$

The solution to the this maximization problem is $q_1 = \frac{3}{8}$, which gives firm 1 a profit of $\frac{9}{64}$.

c. Suppose the oligopoly stage game is repeated infinitely often, and that both firms share discount factor δ. For what range of δ can the firms sustain the collusive outcome identified in a, using grim trigger strategies with the Nash equilibrium as the punishment?

From part b, either player can earn a one-period profit of $\frac{9}{64}$ by deviating. The Nash equilibrium payoffs of the Cournot stage game are $(pi_1, \pi_2) = (\frac{1}{9}, \frac{1}{9})$. From part a, the payoffs under collusion are $(\pi_1, \pi_2) = (\frac{1}{8}, \frac{1}{8})$.

Therefore, collusion supported by a punishment path of Nash reversion is a subgame perfect equilibrium if and only if:

$$\frac{\frac{1}{8}}{1-\delta} \geq \frac{9}{64} + \delta\frac{\frac{1}{9}}{1-\delta}$$
$$\iff \delta \geq \frac{9}{17} \quad (1)$$

Problem 3 Consider the following game:

	L	C	R
U	1,1	21,0	0,0
M	0,21	4,4	0,1
D	0,0	1,0	-1,-1

Suppose the game is repeated infinitely often, with both players sharing discount factor $\delta = .9$. Both players play limited punishment trigger strategies with a punishment phase of T periods of the Nash equilibrium outcome (U,L). Determine the length of the punishment period that is required to support $(4, 4)$ as the payoff in every stage of a subgame perfect equilibrium (how small can we make T such that no player finds it optimal to deviate in any period?).

Consider the case where (U, L) is played for T periods as a punishment for deviations from (M, C). For simplicity, subtract 1 from all of the game's payoffs (this step is unnecessary, but this way the punishment involves each player receiving a payoff of 0 for T periods, which will make payoffs easier to calculate). Then, sticking to the equilibrium path is preferable to a one-period deviation if and only if:

$$\frac{3}{1-\delta} \geq 20 + 0 * (\delta + \delta^2 + ... + \delta^T) + \delta^{T+1}\frac{3}{1-\delta}$$
$$\iff \frac{1}{3} \geq .9^{T+1}$$
$$\iff T \geq 10 \quad (2)$$

Problem 4 Consider an industry that consists of two firms, A and B. They face a demand curve $Q = q_A + q_B = 14 - P$, where P is the industry price of output. Both firms have constant marginal cost of \$2.

a. Suppose they form a cartel and choose the price that maximizes the sum of their profits. Show that they will choose P = \$8.

The monopolist's profit-maximization problem is described as follows:

$$\max_q (14 - q)q - 2q$$

The solution to the problem is $q = 6$, $p = 8$, for a profit of 36. If oligopolists collude, the best they can do is to split the monopoly profits, meaning that each firm earns a profit of 18.

b. Now suppose that instead of forming a cartel, they choose prices simultaneously. If they choose different prices, the firm that chooses the lower price captures the entire market; if they set the same price they split the market evenly. Suppose they play this game once. Show that in a Bertrand equilibrium, both firms will charge \$2.

This is the standard result of the Bertrand model; with equal marginal costs, both firms price at marginal cost. Were there an equilibrium in which the lowest price charged were above marginal cost, the firm charging that price could be profitably undercut by his opponent. Were there an equilibrium in which the lowest price were below marginal cost, that firm would earn a negative profit, and so would prefer to raise its price, even if that meant losing business.

c. Suppose they play this game an infinite number of times. Consider the following grim trigger strategy. Choose the cartel price (i.e., P = \$8) in the first period. Continue to choose the cartel price in subsequent periods if all firms have always chose the cartel price up to that point. If any firm chose a price other than \$8, choose the Bertrand price (i.e., \$2) from that point forward.

For what range of values of the discount factor do these trigger strategies constitute a subgame perfect equilibrium?

While colluding, each firm earns a profit of 18. The most profitable unilateral deviation would be for a firm to slightly undercut its rival by reducing its price by epsilon, where epsilon is an infinitesimal number, and earn a profit of just less than 36 (in practice, we award the 'full' monopoly profit to the deviating firm in this case, but it's ok if you calculated profit if the deviator undercuts his opponent by, say, a penny). Therefore, collusion supported by Nash reversion is a subgame perfect equilibrium of the repeated game if and only if:

$$\frac{18}{1-\delta} \geq 36 + \delta * 0$$
$$\iff \delta \geq \frac{1}{2}$$

d. Now change this game so that there are $N \geq 2$ oligopolists; thus if they all charge the same price, each will sell a proportion $\frac{1}{N}$ of the market demand at that price. Express the critical discount factor (required to obtain an equilibrium in the infinitely repeated game) as a function of N. Does your answer suggest that it will be easier to sustain cooperation when N is small or when N is large? What is the intuition behind this result?

N colluding oligopolists will each earn a stage game profit of $\frac{36}{N}$. Any unilateral deviation will get that firm the full monopoly profit of 36 for one period, followed by the Nash equilibrium payoff of 0 ad infinitum. Therefore, to support collusion as a subgame perfect equilibrium requires:

$$\frac{\frac{36}{N}}{1-\delta} \geq 36$$
$$\iff \delta \geq 1 - \frac{1}{N}$$

In particular, as N increases, the minimum δ required to maintain collusion increases, trending towards 1 as $N \to \infty$. The reason for this is that as the number of firms increases, the share of total profit going to each firm decreases, making a deviation to capture the entire market relatively more tempting.