# Homework 5

**uncollected**

**Problem 1** Suppose that *normal* workers have productivity of \$6, while *smart* workers have productivity of \$A, where $A > 6$. Firms cannot tell smart workers from normal workers *ex ante*, but can observe a worker's education level $e$. Firms know that half of all workers are normal, and half are smart.

Any worker can acquire as much education as she wishes, but getting $e$ units of education costs a normal worker $B * e$, where $B > 1$, and costs a smart worker $e$. Assume the labor market is competitive, so that a worker earns her expected productivity. A worker's lifetime utility function is her wage minus the cost of any education she receives.

**a.** Suppose $A = 20$ and $B = 2$. In a graph with $e$ on the X-axis, and wage on the Y-axis, draw 3 indifference curves for both smart and normal workers. You have enough information for your drawing to be precise.

All graphs appear at the end of this answer sheet. Note that a smart worker's utility is $wage - e$, and so the equation for the indifference curve giving her (say) utility of 20 is $wage - e = 20$. Since we will graph this curve with wage on the y-axis and $e$ on the x-axis, solve for wage: $wage = 20 + e$. The equation for an indifference curve for utility 10 would be $wage = 10 + e$, and so on.

**b.** Suppose $A = 20$ and $B = 2$. Construct a wage function so that there is a pooling equilibrium, with both smart and normal workers obtaining 3 units of education. Describe the wage function you chose using a graph (and, if possible, an equation).

One wage function that would support these education levels as a pooling equilibrium is the following:

$$wage(e) = \begin{cases} 13 & \text{if } e = 3 \\ 6 & \text{if } e \neq 3 \end{cases} \tag{1}$$

See the end of the answer sheet for a picture.

**c.** Use a new graph and a verbal explanation to demonstrate that the equilibrium you constructed in part b does not satisfy the intuitive criterion. Clearly state which part of your wage function fails the criterion.

Consider a worker who gets education $e = 10$. Any wage function which supports $(e_L = e_H = 3)$ as a pooling equilibrium must put at least some weight on a worker with $e = 10$ being a low type, as otherwise the high type of worker would surely prefer to switch from $e = 12$ to $e = 10$ (her utility would increase from $8to10$). However, low types can only be made worse off by choosing $e = 10$, no matter what the wage is. Even if a low type is paid the maximum wage of 20 (the productivity of a high type), his utility would be only $20 - 2 * 10 = 0$, whereas he gets utility 5 from choosing 3 units of education. Therefore, the intuitive criterion requires firms to hold belief $\mu(H|e = 10) = 1$, in which case they must pay a wage of 20 for any worker who chooses 10 units of education. This breaks the equilibrium identified in part b.

**d.** Suppose that $A = 20$ and $B = 2$. Construct a wage function so that there a separating equilibrium in which normal types get education $e_N = 0$, while smart types gets $e_S = 10$. Depict the equilibrium graphically.

One wage function that would support these education levels as a separating equilibrium is the following:

$$wage(e) = \begin{cases} 20 & \text{if } e = 10 \\ 6 & \text{if } e \neq 10 \end{cases} \tag{2}$$

See the end of the answer sheet for a picture.

**e.** Use a new graph and a verbal explanation to demonstrate that the equilibrium you constructed in part d does not satisfy the intuitive criterion. Clearly state which part of your wage function fails the criterion.

Consider a worker who chooses $e = 8$. Any wage function that is part of an equilibrium must pay $wage(8) < 20$, otherwise the high type of worker would surely choose to switch to $e = 8$. But the intuitive criterion says that a worker choosing $e = 8$ must be a high type, since the low type of worker could only be made worse off relative to his equilibrium payoff by choosing $e = 8$. The low type gets utility of 6 in the equilibrium of part d, yet even if he were paid the maximum wage of 20, would get utility of only 4 from choosing $e = 8$. Therefore, the intuitive criterion requires $wage(e = 8) = 20$, which breaks the equilibrium in part d.

**f.** Describe, using a graph and words, the unique equilibrium outcome $(e_N, e_S)$ of this game that satisfies the intuitive criterion.

The unique equilibrium satisfying the intuitive criterion is a separating equilibrium where the high types get just enough education to leave the low types indifferent between switching to $e_H$ and staying at $e_L = 0$. Since a choice of $e = 0$ gives a low type utility 6, a choice of $e = 7$ and a wage of 20 would give the low type the same utility. Hence, the unique equilibrium outcome is $e_L = 0, e_H = 7$, and this is supported by a wage function such as the following:

$$wage(e) = \begin{cases} 20 & \text{if } e = 7 \\ 6 & \text{if } e \neq 7 \end{cases} \tag{3}$$

**g.** For general values of $A$ and $B$, determine the unique equilibrium outcome $(e_N, e_S)$ satisfying the intuitive criterion.

To leave the low type indifferent between switching and not, $e_H$ must satisfy $6 = A - be_H$, so $e_H = \frac{A-6}{b}$.

**h.** Explain verbally how the outcome in g is affected by an increase in $A$. Explain intuitively why this is the case. Do the same for an increase in $B$.

$e_H$ is increasing in $A$ and decreasing in $b$. The intuition is that as $A$ increases, imitating a high type becomes relatively more valuable, and so high types must choose a higher level of education to deter the low types from choosing $e_H$. As $b$ increases, education becomes more costly for the low types, and so the high types do not need to choose as high of a level of $e_H$ to deter the low types from imitating the high types.

**Problem 2** Suppose that *low-ability* workers have productivity of $D$, while *high-ability* workers have productivity of $A$, where $A > D$. Firms cannot tell low-ability workers from high-ability workers *ex ante*, but can observe a worker's education level $e$. Firms know that half of all workers are low-ability, and half are high-ability.

Any worker can acquire as much education as she wishes, but getting $e$ units of education costs a low-ability worker $B * e$, where $B > 1$, and costs a high-ability worker $e$. Assume the labor market is competitive, so that a worker earns her expected productivity.

**a.** Suppose $A = 15$, $B = 4$, and $D = 1$. Does there exist a pooling equilibrium in which both high- and low-ability workers get 1 unit of education? If so, draw or describe a wage function that supports this equilibrium outcome, and determine whether or not the equilibrium satisfies the intuitive criterion. If not, explain why not.

If all workers pool on $e = 1$, they would earn a wage of 8, and so high types would receive utility of 7, and low types utility of 4. The wage function that is most favorable to this being a pooling equilibrium is $w(1) = 8$ and $w(e) = 1$ for all $e \neq 1$. Under this wage function, the most profitable deviation for either type of worker is to $e = 0$. Such a deviation would give both high and low types utility of 1, which is less than their equilibrium utility. Conclude that there is a pooling equilibrium at $e = 1$, supported by the wage function described here. No pooling equilibrium satisfies the intuitive criterion. The listed wage function fails the intuitive criterion, because (for example) the wage function implies firms believe that a worker with 3 units of education is certainly a low type, but the maximal utility a low type can achieve with 3 units of education is $15 - 3 * 4 = 3$, which is less than their equilibrium payoff of 4.

**b.** Suppose $A = 15$, $B = 4$, and $D = 1$. Does there exist a pooling equilibrium in which both high- and low-ability workers get 3 units of education? If so, draw or describe a wage function that supports this equilibrium outcome, and determine whether or not the equilibrium satisfies the intuitive criterion. If not, explain why not.

In this case, low types receive utility of -4, and so would prefer to deviate to $e = 0$, where they would get at least 1 unit of utility. No matter the wage function, there is no equilibrium in which workers pool at $e = 3$.

**c.** Suppose $A = 15$, $B = 4$, and $D = 1$. Solve for a separating equilibrium which *does not* satisfy the intuitive criterion. Draw or describe a wage function that supports this outcome in an equilibrium. Clearly explain, using your graph and/or a verbal description, why this equilibrium fails the intuitive criterion.

From the answer to part d., in the unique equilibrium satisfying the intuitive criterion, high types choose $e = 3.5$.

Consider, then, the separating equilibrium in which $e_H = 4$, $e_L = 0$, $w(4) = 15$, and $w(e) = 1$ for all $e \neq 4$. Both types are optimizing and the firm has correct beliefs, so this is an equilibrium. However, the firm pays a wage of 1 in response to $e = 3.6$, and thus must believe that a worker acquiring 3.6 units of education is a low type with probability 1. However, any low type who acquires more than 3.5 units of education receives a utility of less than 1. Since 1 is the low type's equilibrium payoff, the firm's belief is disallowed by the intuitive criterion.

**d.** For general $A$, $B$, and $D$, solve for the unique equilibrium which *does* satisfy the intuitive criterion as a function of $A$, $B$ and $D$. How does the level of education obtained by the high types vary in $D$ in this equilibrium? What is the intuition?

In the intuitive criterion equilibrium, high types choose $e_H$ units of education, where $e_H$ leaves low types indifferent between $e_L = 0$ and $e_L = e_H$, or $D = A - B * e_H$, or $e_H = \frac{A-D}{B}$. As $D$ increases, the equilibrium value of $e_H$ decreases; this is because less education is needed to separate from low types, who are now paid a higher wage since they are more productive.

**Problem 3** This problem asks you to consider an extension of the basic Spence model to one in which education is productive and the cost of education is convex.

Suppose that high types with education $e$ have productivity $y(H, e) = 10 + 2e$, while low types have productivity $y(L, e) = 2 + e$. Firms cannot observe whether a worker is a high type or a low type, but know that half of all workers are of each type. A competitive labor market ensures each type of worker is paid her expected productivity. A high type can acquire $e$ units of education at cost $c_H(e) = \frac{1}{10}e^2$, while education costs a low type $c_L(e) = \frac{1}{4}(e+2)^2 - 1$.

**a.** Suppose $e_L = 0$ and $e_H = 12$. What wage function would support this outcome as a separating equilibrium? Draw a picture and/or describe using an equation.

Consider the following wage function:

$$wage(e) = \begin{cases} 34 & \text{if } e = 12 \\ 2+e & \text{if } e \neq 12 \end{cases} \tag{4}$$

High types will maximize their utility by choosing $e = 12$ (giving utility of 19.6), while low types will maximize their utility by choosing $e = 0$ (giving utility of 2, versus $-14$ from deviating to $e = 12$).

**b.** Does the equilibrium you described in part a satisfy the intuitive criterion? Why or why not?

No. Consider education level $e = 10$. If $wage(e = 10) = 30$, high types would prefer $e = 10$ to $e = 12$, since their utility would increase to 20. Low types would not like to switch to $e = 10$ from $e = 0$ regardless of how high the wage is, since even if $wage(10) = 30$, low types utility from choosing $e = 10$ would be only $-5$.

**c.** Draw the set of all points which give the high type utility of 5. What is the slope of the indifference curve you drew, as a function of $e$? Determine the point of tangency between the high type's indifference curve and the function $y(H, e)$ (note that the high type may get more or less than 5 utility at the point of tangency). Do the same for the low type's indifference curve and the function $y(L, e)$.

The slope of the high type's indifference curve is $\frac{1}{5}e$, while the slope of her productivity function is 2. The point of tangency is then at $e = 10$. For the low types, the slope of the indifference curve is $\frac{1}{2}(e+2)$, while the slope of his productivity function is 1, meaning to point of tangency is at $e = 0$.

**d.** Suppose that both types choose their education level so that their indifference curve is tangent to their productivity function. Describe, using a picture and/or an equation, a wage function that would support this outcome as a perfect Bayesian equilibrium.

Consider the following wage function:

$$wage(e) = \begin{cases} 30 & \text{if } e = 10 \\ 2+e & \text{if } e \neq 10 \end{cases} \tag{5}$$

Low types strictly prefer $e = 0$ to all other education levels, and high types strictly prefer $e = 10$ to all other education levels (since they are by definition on their highest achievable indifference curve over all the points on their productivity function $10 + 2e$).

**e.** Does the equilibrium you described in part d satisfy the intuitive criterion? Why or why not?

Yes. Any education level other than $e = 10$ makes the high type worse off, so the intuitive criterion has no bite. (There are education levels that could potentially make only the low type better off, such as $e = 1$, but requiring $\mu(L|e = 1) = 1$ does not affect the equilibrium outcome; indeed, exactly this belief is embedded in the wage function described in part d.).

**Problem 4** Consider the game in Figure 1 below.

**a.** Draw the reduced normal form. Find all pure strategy Nash equilibria. There is a mixed Nash equilibrium in which 1 randomizes between A and B, and 2 randomizes between L and R. Find it.

One pure Nash equilibrium, at $(C, M)$. The mixed Nash equilibrium is at $(\frac{1}{2}A + \frac{1}{2}B, \frac{1}{2}L + \frac{1}{2}R)$.

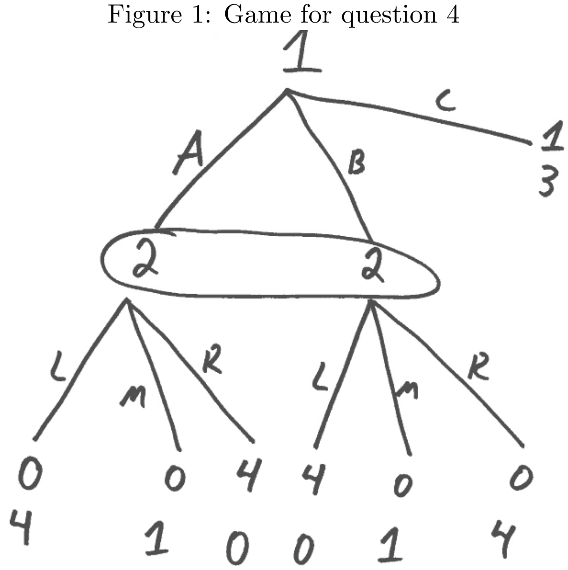
**b.** Find all of the game's perfect Bayesian equilibria (pure as well as mixed).

The one PBE is $(\frac{1}{2}A + \frac{1}{2}B, \frac{1}{2}L + \frac{1}{2}R)$, with 2 believing that each node is equally likely.

**c.** Explain in intuitive terms any differences between your answers to part a and part b.

The pure Nash equilibrium involves 2 playing a strictly dominated strategy. PBE rules this out.

Figure 1: Game for question 4




**Problem 5** A firm has two types of jobs, good jobs and bad jobs. When a *qualified* worker is assigned to a good job, the firm earns a net profit of \$20,000. When an *unqualified* worker is assigned to a good job, the firm incurs a net loss of \$20,000. When a worker of either type is assigned to a bad job, the firm breaks even. Workers prefer good jobs, and get an extra \$32,000 payoff from a good job relative to a bad job.

To become qualified, a worker pays an investment cost $c$. This cost is higher for some workers than for others; the distribution of $c$ across all workers is uniform between \$0 and \$9,000. The firm cannot observe which workers are qualified and which are not.

Suppose that while the firm cannot directly observe workers' investment decisions, it administers a test to new employees, with scores ranging from 0 to 1. The probability a qualified worker scores less than $t \in [0, 1]$ is $t^2$. The probability an unqualified worker scores less than $t$ is $t$.

**a.** Suppose that the firm puts all workers with a test score of $s \in [0, 1]$ or higher into a good job. Describe the incentive constraint for a worker's decision to become qualified or not. What fraction $\pi$ of workers will become qualified, as a function of $s$?

The benefit to becoming qualified is $32,000(s - s^2)$. The cost is $c$. Given $c \sim U[0, 9000]$, the fraction of workers who become qualified, as a function of $s$, is

$$\pi = \frac{32}{9} s(1 - s) \tag{6}$$

**b.** Now consider the firm's problem. Suppose that fraction $\pi$ of all workers become qualified, so that the firm's prior is $\pi$. Show that the firm optimally puts workers scoring above some cutoff test score $s$ into good jobs, and puts low-scoring workers into bad jobs, and solve for $s$ as a function of $\pi$.

The firm's posterior belief that a given worker who received test score $\theta$ is qualified is $p(\theta) = \frac{\pi 2\theta}{\pi 2\theta + 1 - \pi}$. The firm will put the worker into the good job iff $p * 20,000 - (1 - p)20,000 \geq 0$. Simplifying, the firm will put a worker into a good job iff $\pi \geq \frac{1}{1+2\theta}$. The firm's cutoff test score is determined by where this holds with equality, or

$$s = \frac{1 - \pi}{2\pi} \tag{7}$$

**c.** An equilibrium is $(s, \pi)$ pair such that $s$ maximizes firm profit given $\pi$ and $\pi$ is consistent with workers maximizing expected wages net of the investment cost given $s$. Characterize the equilibrium values of $\pi$ and $s$ as follows. One, show that $s = \frac{1}{4}$ and $\pi = \frac{2}{3}$ is an equilibrium. Two, show with a picture that there is another equilibrium with $s > \frac{1}{4}$ (you may solve for this equilibrium using a computer if you like, but the answer may not involve round numbers).

The pair $(\pi, s) = (\frac{2}{3}, \frac{1}{4})$ clearly satisfies both (6) and (7), as required for an equilibrium. A picture helps to show that there is a second equilibrium. See the figure at the end of this answer set. Equation (6) is a concave function maximizes at $s = \frac{1}{2}$ and equal to zero at $s = 0$ and $s = 1$. Equation (7) is a downward sloping function that crosses the red line at $s = \frac{1}{4}$ and is always positive. This implies that there must be a second equilibrium at $s_2^* > \frac{1}{4}$.

**d.** What economic interpretation does the Coate and Loury paper studied in class assign to the multiplicity of equilibria in its model?

The possibility of rational discrimination; "bad" equilibria correspond to discriminated-against groups, and "good" equilibria correspond to favored groups.

**Problem 6** Consider an economy in which there are equal numbers of two kinds of workers, $A$ and $B$, and two kinds of jobs, good and bad. Each employer has an unlimited number of vacancies in both kinds of jobs. Some workers are qualified for the good job, and some are not. If a qualified worker is assigned to the good job the employer gains \$2,000, and if an unqualified worker is assigned to the good job the employer loses \$1,000. When any worker is assigned to the bad job, the employer breaks even.

Workers who apply for jobs are tested and assigned to the good job if they do well on the test. Test scores range from 0 to 1. The probability that a qualified worker will have a test score less than $\theta$ is $\theta^2$.

The probability that an unqualified worker will have a test score less than $\theta$ is $\theta$. These probabilities are the same for A-workers and B-workers.

There is a fixed wage premium of $4,000 attached to the good job. Workers can become qualified by paying an investment cost, and this cost is higher for some workers than for others: the distribution of costs is uniform between 0 and $3,000, for both A-workers and B-workers.

Workers make investment decisions so as to maximize earnings, net of the investment cost (all of these amounts are expressed as present values).

Can you find an equilibrium in which there are more A-workers than B-workers in the good jobs?

Using the notation from class, we have:

$$x_q = 2000$$
$$x_u = 1000$$
$$F_q(\theta) = \theta^2$$
$$f_q(\theta) = 2\theta$$
$$F_u(\theta) = \theta$$
$$f_u(\theta) = 1$$
$$\omega = 4000$$
$$G(c) = \frac{c}{3000} \quad \text{for } c \in [0, 3000]$$

The EE and WW equations are then given by:

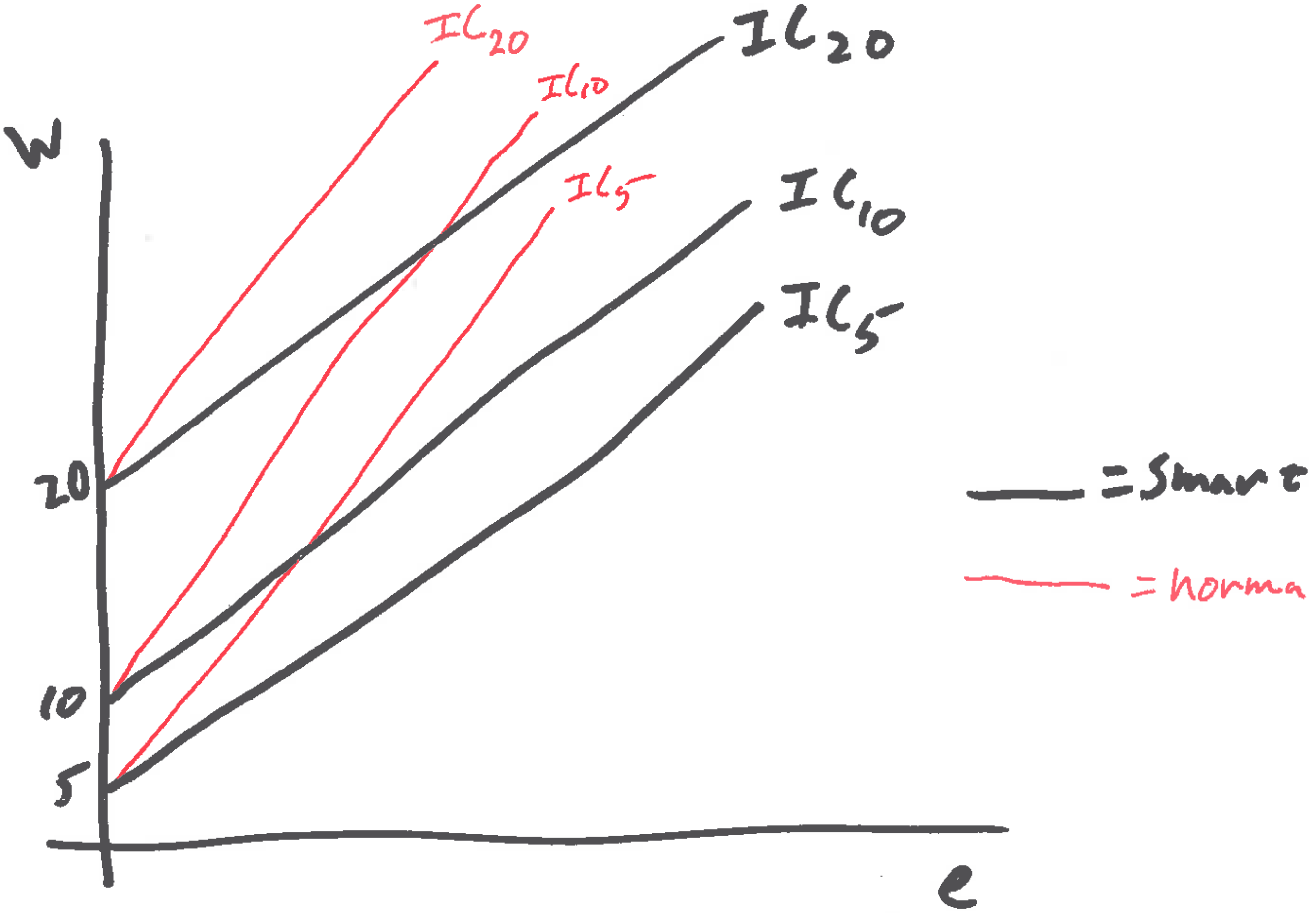
$$2 = \frac{1-\pi}{\pi}\frac{1}{2s} \quad \text{(EE)}$$
$$\pi = \frac{4}{3}s(1-s) \quad \text{(WW)}$$

Solving (EE) and (WW) for $\pi$ and $s$ yields two solutions in which both $\pi$ and $s$ are in $[0, 1]$:

$$\text{Solution 1: } s = \frac{3}{4}, \pi = \frac{1}{4}$$
$$\text{Solution 2: } s = \frac{1}{2}, \pi = \frac{1}{3}$$

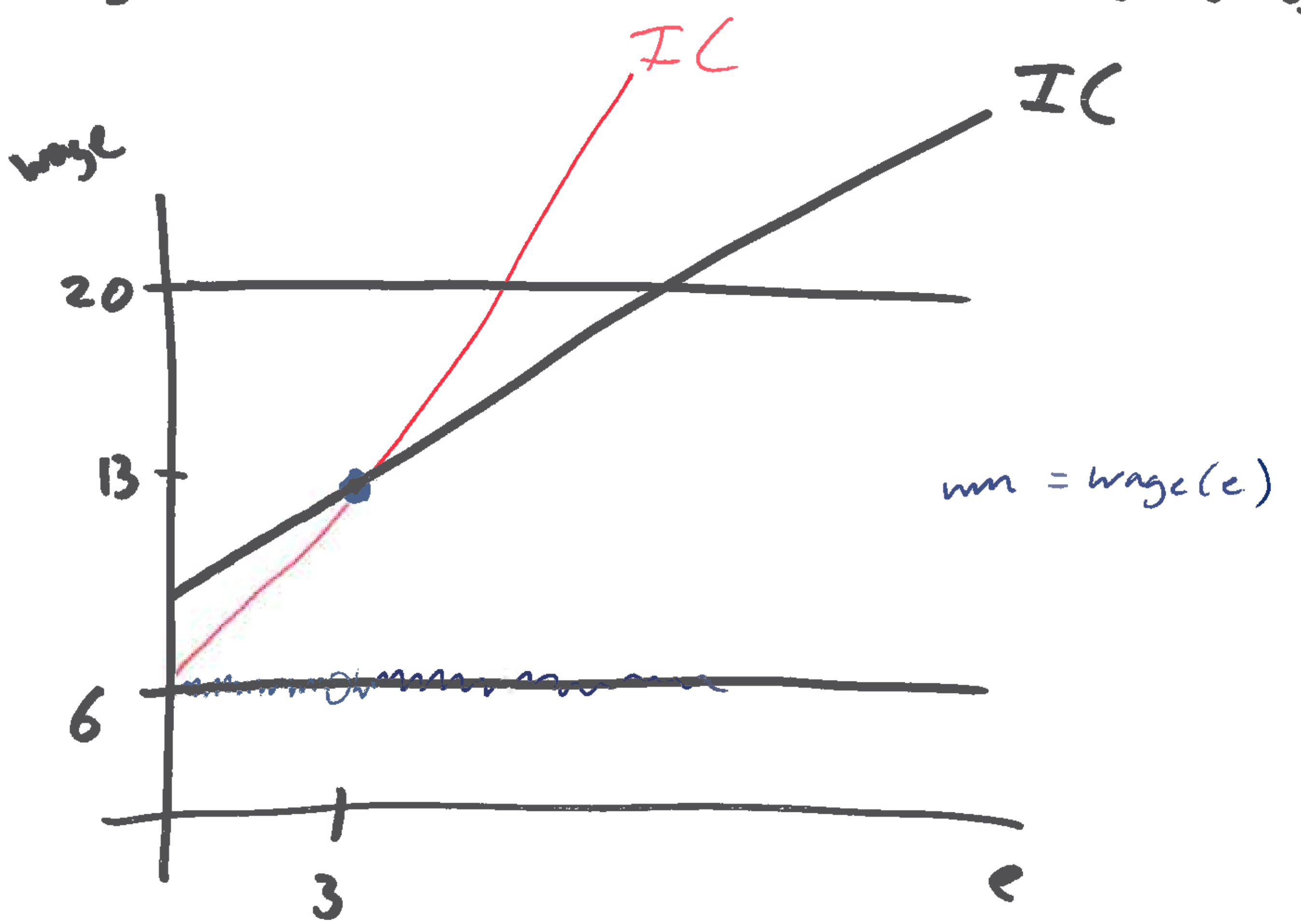
If the $(s = \frac{3}{4}, \pi = 14)$ equilibrium is applied to B-workers, while the $(s = \frac{1}{2}, \pi = 13)$ equilibrium is applied to A-workers, then, despite A- and B-workers being ex ante identical, statistical discrimination against B-workers will persist in equilibrium.

1a

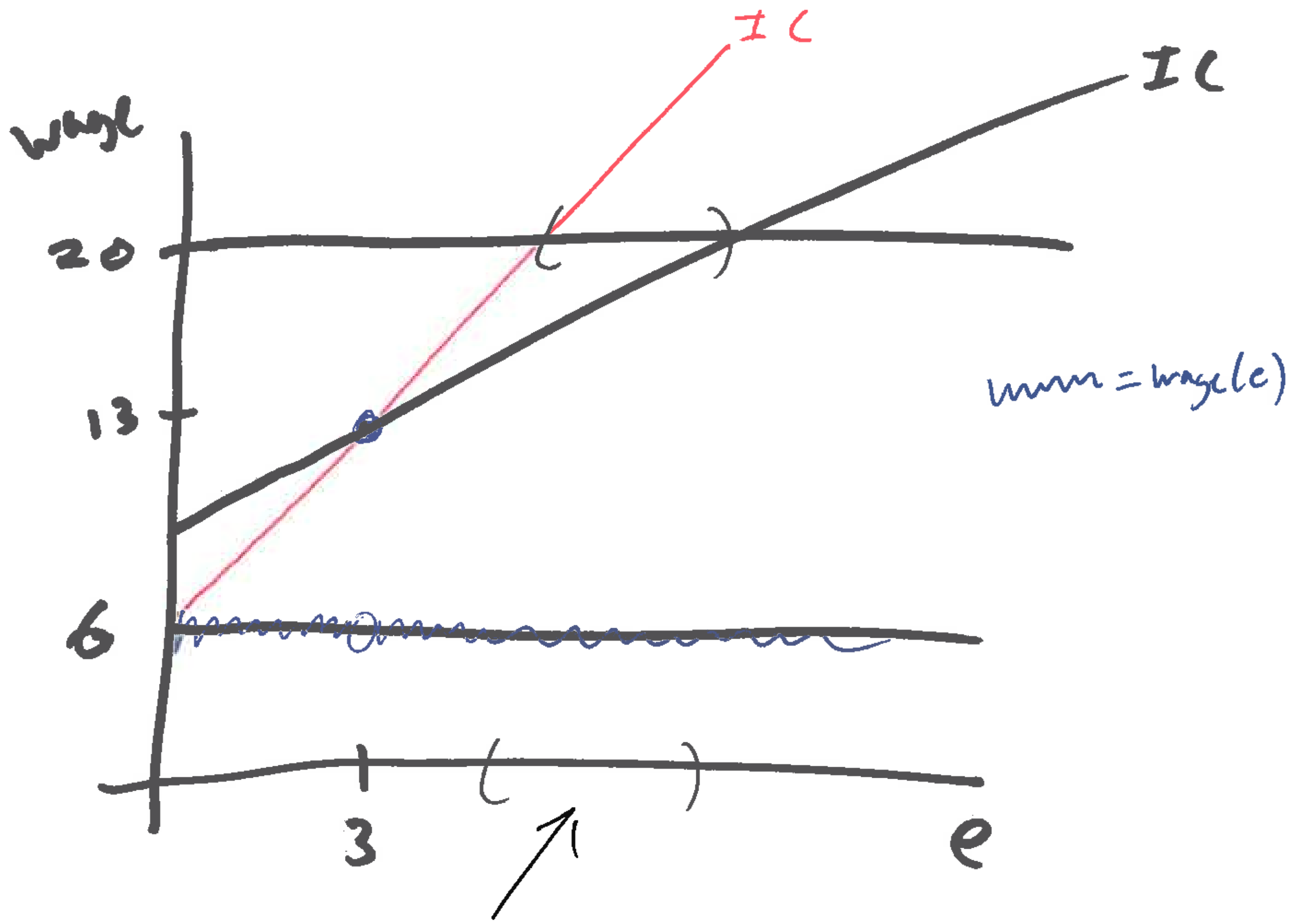



Smart workers' indifference curves

1b.

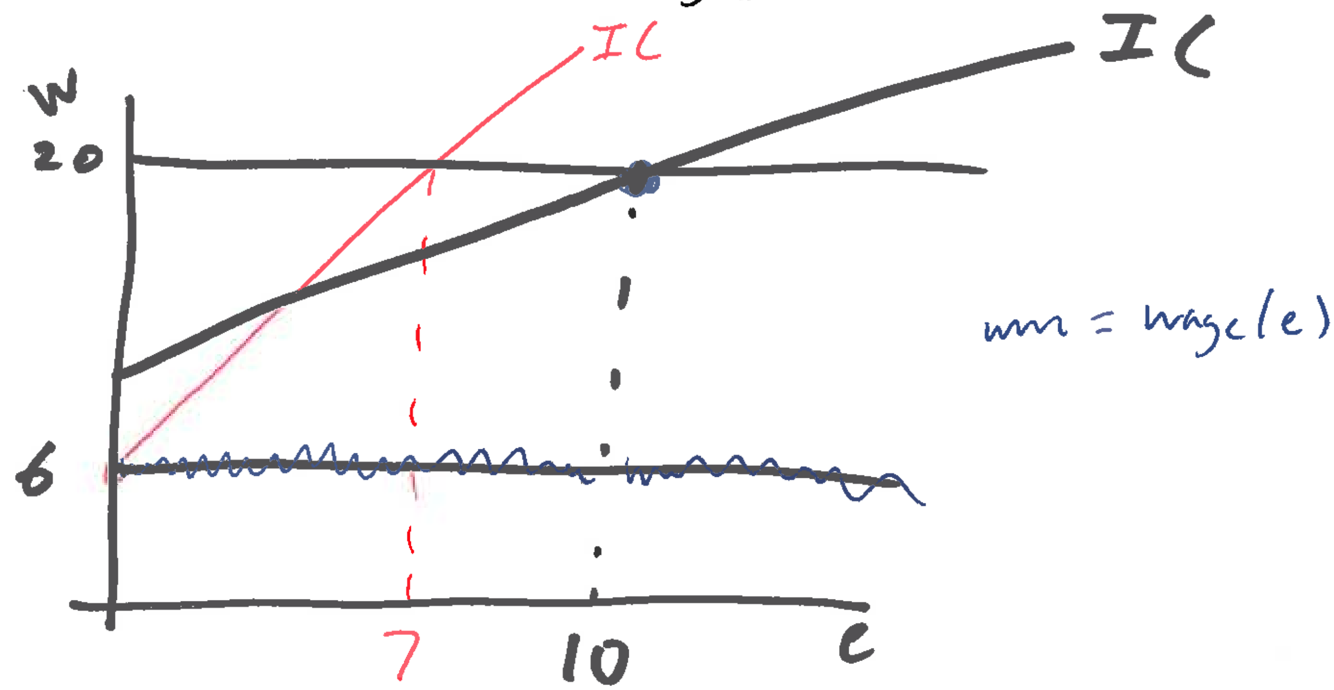


1c.

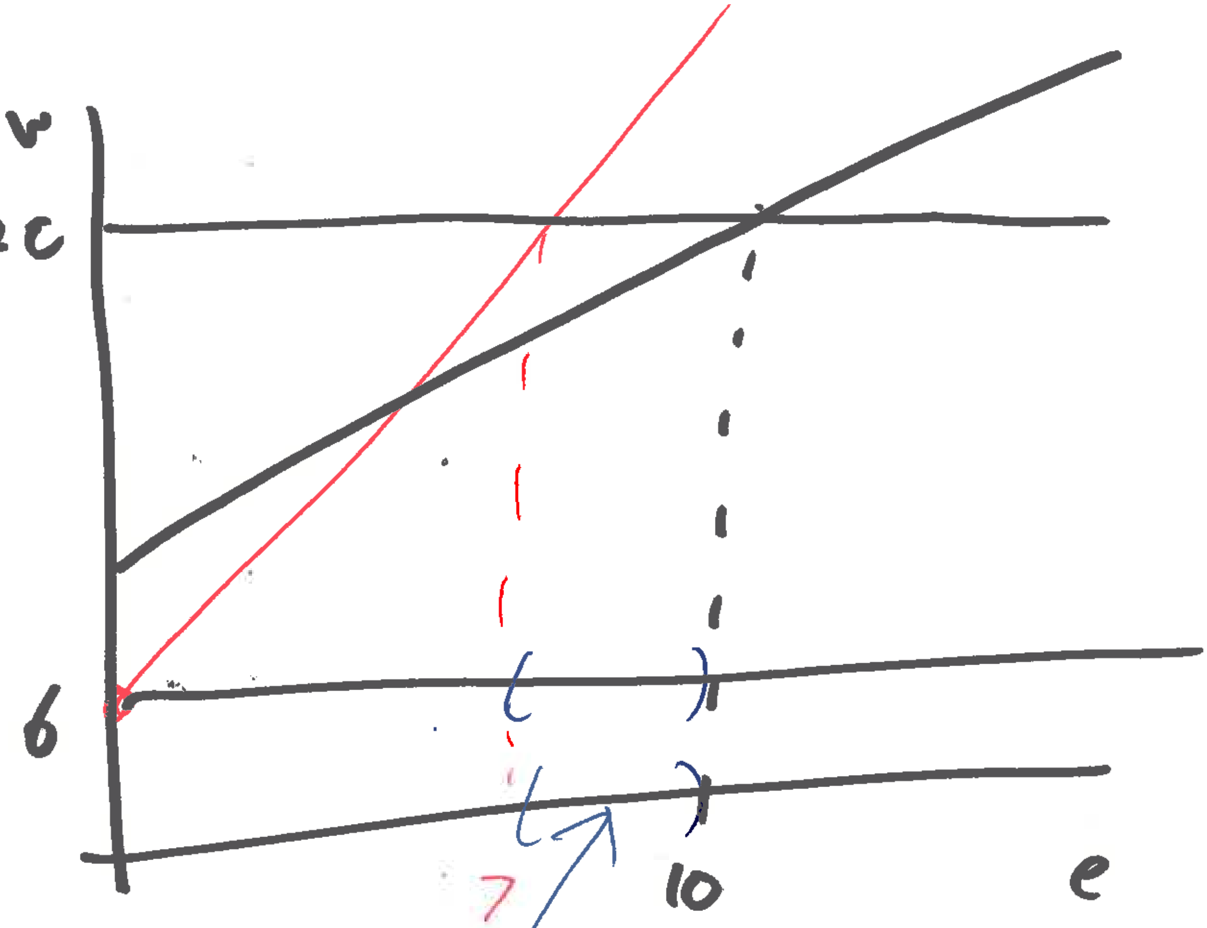



For any education level in this range, the firm should pay a wage of 20, as only the smart workers would possibly choose a value of e in this range.

1d.

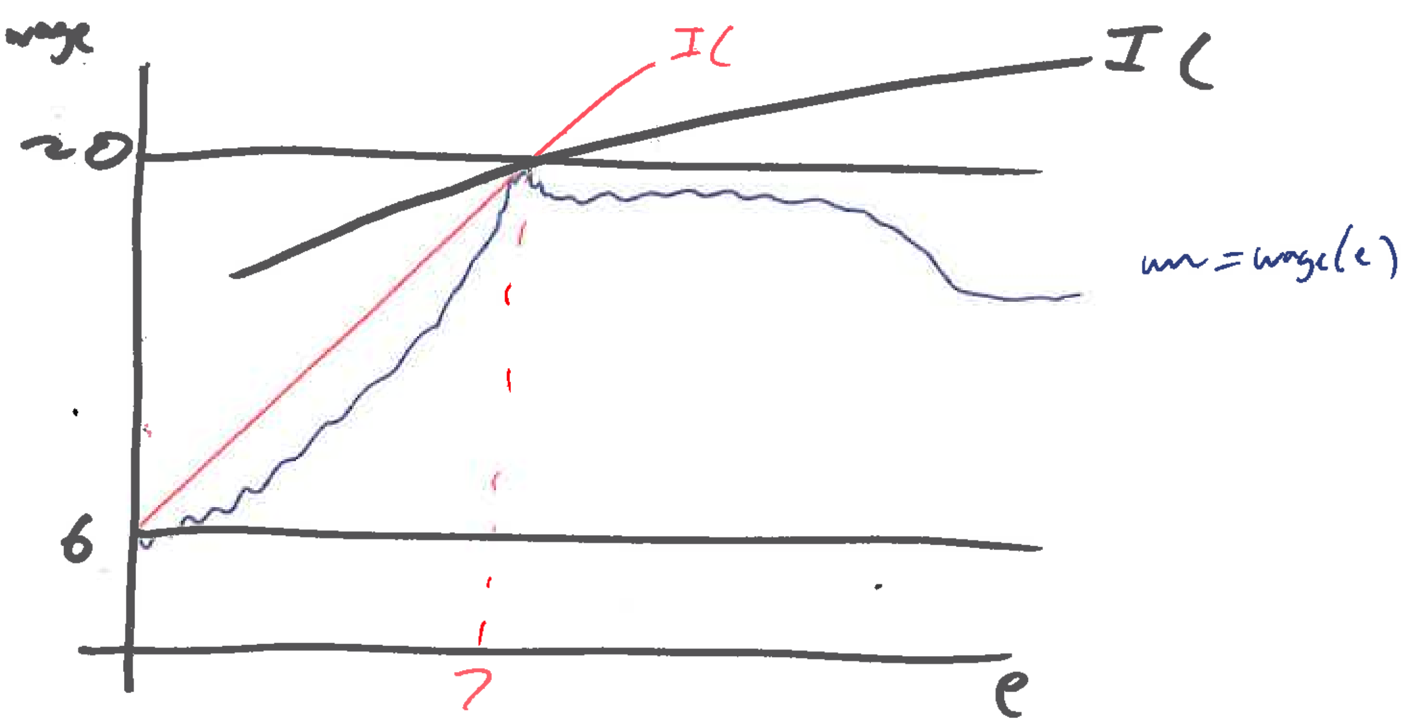


1e




For any education level between 7 and 10, the firm should recognize that only the ~~high~~ smart type is potentially better off, and thus should pay a wage of 20.

1f.



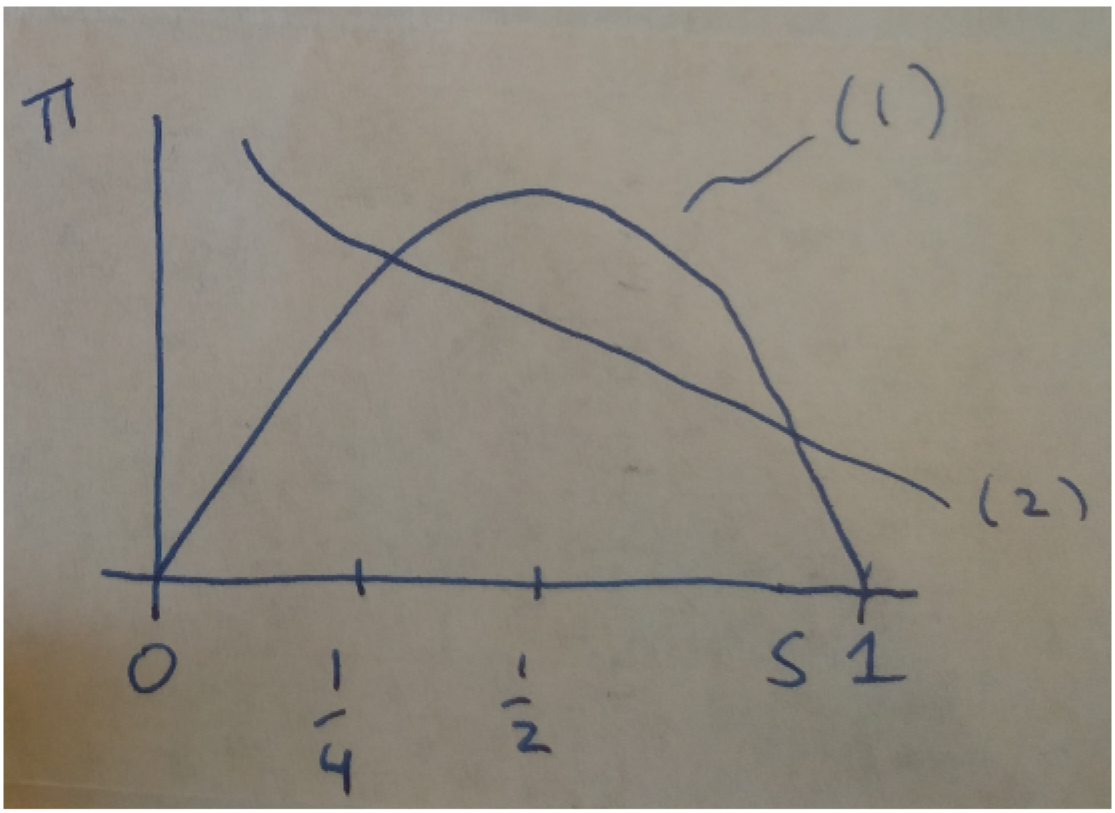



Figure 2: Figure mentioned in the answer to 5c.