Homework 1

answers

Problem 1 Kirt and Lila are engaged in a joint project. If person $i \in \{K, L\}$ invests effort $x_i \in [0, 1]$ in the project, at cost $c(x_i)$, the outcome of the project is worth $f(x_K, x_L)$. The worth of the project is split equally by Kirt and Lila, regardless of their effort levels, so that each gets a payoff of $\frac{1}{2}f(x_K, x_L) - c(x_i)$. Suppose effort levels are chosen simultaneously.

a. Suppose $f(x_K, x_L) = 3x_K x_L$ and that $c(x_i) = x_i^2$. Find the Nash equilibrium effort levels of this simultaneous move game.

K solves the following, taking x_L as given: $\max_{x_K} \frac{3}{2}x_K x_L - x_K^2$, which has a maximum located at $x_K^* = \frac{3}{4}x_L$. Similarly, L's payoff is maximized at $x_L^* = \frac{3}{4}x_L^*$. The only values that satisfy both of these equations are $(x_K^*, x_L^*) = (0, 0)$. Therefore, the unique Nash equilibrium of this game is located at $(0, 0)$.

b. Is there a pair of effort levels that yield higher payoffs for both players than do the Nash equilibrium effort levels in part a.?

In the Nash equilibrium described in part a, both players get a payoff of 0. Now consider the alternative arrangement $x_K = x_L = 1$. Here, each player gets a payoff of $\frac{1}{2}$, and is therefore better off than in the Nash equilibrium.

Problem 2 Consider the normal form game below:

		Avon I	Avon N
Joe	I	r, r	$r-1, 0$
Joe	N	$0, r-1$	$0, 0$

In this game, strategy I represents investing, and strategy N represents not investing. Investing yields a payoff of r or $r - 1$, according to whether the player's opponent invests or not. Not investing yields a certain payoff of 0.

Describe the set of Nash equilibria (pure and mixed) of the game for each $r \in [-2, 3]$.

First, if $r \in (1, 3]$, strategy I is dominant for both players, and so r, r is the unique Nash equibrium. Likewise, if $r \in [-2, 0)$, N is strictly dominant for each player, and so N, N is the unique Nash equilibrium. If $r \in [0, 1]$, then both (I, I), (N, N), and $((1 - r)I + rN, (1 - r)I + rN)$ are Nash equilibria, with the last being a mixed equilibrium in which each player plays I with probability $1 - r$.

Problem 3 Gibbons, problem 1.13

This is similar to the Hawk-Dove game studied in class. There are two pure-strategy Nash equilibria, in which the workers apply to different jobs (1 applies to firm 1, 2 applies to firm 2; 1 applies to firm 2, 2 applies to firm 1). There is also a mixed strategy Nash equilibrium, in which each worker applies to firm 1 with probability $\frac{2w_1 - w_2}{w_1 + w_2}$ and to firm 2 with probability $\frac{2w_2 - w_1}{w_1 + w_2}$. It is simple to verify that each of these probabilities is between 0 and 1.

Problem 4 Three firms are considering entering a new market. The total profit obtained by each firm depends on the number of firms that enter. If all three firms enter, each firm loses \$50. If two firms enter, each firm makes \$10. If only one firm enters, that firm makes \$30. Assume entry is costless, and that any firm that does not enter receives a payoff of 0.

a. Find all pure strategy Nash equilibria. There are three: 1- Firms 1 and 2 enter, firm 3 does not. 2- Firms 1 and 3 enter, firm 2 does not. 3- Firms 2 and 3 enter, firm 1 does not.

b. Find the symmetric mixed-strategy equilibrium in which all three firms enter with the same probability p. In expectation, what profit does each firm earn in this equilibrium? In such an equilibrium, each firm is indifferent between entering and not, meaning that the expected value of entering is 0. If both rivals enter with probability p, then a firm's expected payoff from entering is:

$$\begin{aligned}\pi(enter) =& p^2 * (-50) + 2p(1-p) * 10 + (1-p)^2 * 30 = 0 \\ \iff& 40p^2 + 40p - 30 = 0 \\ \Rightarrow& p = \frac{1}{2}\end{aligned}$$

This indifference condition implies that in the unique symmetric equilibrium, each firm enters with probability $\frac{1}{2}$.

Problem 5 Two duopolists producing differentiated goods face the following demand system:

$$q_1^D = 10 - p_1 + \frac{1}{2}p_2$$
$$q_2^D = 10 - p_2 + \frac{1}{2}p_1$$

Each firm has marginal cost 0. The two firms simultaneously set price and then sell quantity $q_i^D(p_1, p_2)$, $i = 1, 2$.

a. Solve for the Nash equilibrium prices. What profit does each firm earn?

The Nash equilibrium is $p_1 = p_2 = \frac{20}{3}$, $\pi_1 = \pi_2 = \frac{400}{9}$.

b. Now suppose the firms merge. The demand system remains the same, but now one firm jointly sets both p_1 and p_2. Solve for the merged firm's optimal prices and resulting profit.

The merged firm sets $p_1 = p_2 = 10$, for a total profit of 100.

Now suppose each firm has a capacity constraint K_i, such that its marginal cost becomes prohibitively high above K_i. For example, K_i may be the capacity of the firm's factory.

c. Suppose $K_1 = K_2 = 6$. Show that in the pre-merger Nash equilibrium, each firm sets a price of \$8. Solve for the prices the merged firm would set following a merger.

Since each firm has a capacity less than the Nash equilibrium quantity of $\frac{20}{3}$, each will increase price until its demand equals its capacity, conditional on its rival's price. Hence, the equations determining price are:

$$6 = 10 - p_1 + \frac{1}{2}p_2$$
$$6 = 10 - p_2 + \frac{1}{2}p_1$$

Jointly solving yields $p_1 = p_2 = 8$. Post-merger, we know from part b. that the merged firm would be unconstrained at its optimal price vector, hence the constraint is irrelevant and it sets $p_1 = p_2 = 10$.

d. Suppose $K_1 = K_2 = 4$. Solve for the pre-merger Nash equilibrium prices and the optimal post-merger prices.

From parts a. and b., here the constraint will bind both before and after the merger, so in both cases prices are determined by:

$$4 = 10 - p_1 + \frac{1}{2}p_2$$
$$4 = 10 - p_2 + \frac{1}{2}p_1$$

Solving, we have that $p_1 = p_2 = 12$ both before and after the merger.

e. Are merger price effects affected by binding pre-merger capacity constraints?

Merger price effects when firms are capacity-constrained depends on how tightly the constraints bind. Post-merger, prices may or may not increase.

Problem 6 99 shepherds share a common field in which they graze their sheep. Each shepherd purchases as many sheep as he/she likes, at a cost of $c = \$300$/sheep. The value of one sheep is given by:

$$v(G) = 2000 - S$$

where S is the total number of sheep which graze in the field (more sheep mean less grass/sheep, more sheep fights, etc). The common field is the only suitable location for grazing, and sheep die without grazing, so you may assume that all purchased sheep are brought to graze in the field.

a. In a symmetric Nash equilibrium, how many sheep does each shepherd purchase? How much profit is earned by each shepherd? Each shepherd will purchase 17 sheep, and earn a profit of $289.

b. What is the socially optimal number of sheep? If the resulting total profit is split evenly amongst all shepherds, what is the profit for each shepherd?

In a social optimum, there are 850 total sheep, or 8.58/shepherd. If the total profit from 850 sheep is split among the 99 shepherds, each will earn $7,297.98.

c. Suppose a government imposes a tax on sheep of $T/head, but that the revenue collected from the tax is distributed evenly to each of the 99 shepherds, regardless of how many sheep the shepherd owns. Is such a tax always welfare-reducing? Why or why not?

Such a tax can internalize the externality created when a shepherd purchases an additional sheep (namely, that the sheep lower the value of the other 98 shepherds' sheep). Under a tax, each shepherd earns a profit of $(2000 - S) * s_i - (300 + T) * s_i + T * S/99$ (note that the last term $T * S/99$, represents the amount of tax paid that will be rebated to each shepherd by the government. Alternatively, we could ignore this term, reasoning that an individual does not believe she will receive additional services from the government upon paying extra taxes herself.) In the symmetric Nash equilibrium, each shepherd chooses $s = 17 - \frac{T}{100} + \frac{T}{99*100}$ sheep (note that the last term is quite small). Suppose that $T = 100$. Then, each shepherd will purchase about 16 sheep. Thus, the gross value to owning a sheep will be about 416, and so each sheep will earn a profit of \$16, so that each shepherd's profit from sheep rearing is $16 * 16 = \$256$. Then, the government collects $16 * 99 * \$100 = \$158,400$ in tax revenue. If it distributes this revenue evenly to the 99 shepherds, each will get $1,600 in transfers, and so clearly each shepherd will be better off with the tax than without.

Problem 7 Refer to the Games 1 and 2 depicted in Figures 1 and 2.

a. Enter payoffs into the games below so that they have each of the following properties:

- The games are equivalent except for the addition of strategy C to game 2 (i.e. the first two rows of Game 1 are equivalent to the first two rows of Game 2).
- The only Nash equilibrium of Game 1 is (A, L).
- The only Nash equilibrium of Game 2 is (C, R).

There are many correct answers. One is given below.

Player 1 \ Player 2	L	R
A	3,3	0,0
B	0,0	-1,0

Figure 1: Game 1

Player 1 \ Player 2	L	R
A	3,3	0,0
B	0,0	-1,0
C	4,0	2,2

Figure 2: Game 2

b. Referring only to Game 1, enter payoffs such taht the only Nash equilibrium is both players mixing with equal probability on each strategy.

Again, there are many possible correct answers. One example is given below.

Player 1 \ Player 2	L	R
A	3,0	0,3
B	0,3	3,0