Homework 5

due April 18, 2019

Problem 1 Suppose that *normal* workers have productivity of \$6, while *smart* workers have productivity of \$A, where $A > 6$. Firms cannot tell smart workers from normal workers *ex ante*, but can observe a worker's education level e. Firms know that half of all workers are normal, and half are smart.

Any worker can acquire as much education as she wishes, but getting e units of education costs a normal worker $B * e$, where $B > 1$, and costs a smart worker e. Assume the labor market is competitive, so that a worker earns her expected productivity. A worker's lifetime utility function is her wage minus the cost of any education she receives.

a. Suppose $A = 20$ and $B = 2$. In a graph with e on the X-axis, and wage on the Y-axis, draw 3 indifference curves for both smart and normal workers. You have enough information for your drawing to be precise.

All graphs appear at the end of this answer sheet. Note that a smart worker's utility is $wage - e$, and so the equation for the indifference curve giving her (say) utility of 20 is $wage - e = 20$. Since we will graph this curve with wage on the y-axis and e on the x-axis, solve for wage: $wage = 20 + e$. The equation for an indifference curve for utility 10 would be $wage = 10 + e$, and so on.

b. Suppose $A = 20$ and $B = 2$. Construct a wage function so that there is a pooling equilibrium, with both smart and normal workers obtaining 3 units of education. Describe the wage function you chose using a graph (and, if possible, an equation).

One wage function that would support these education levels as a pooling equilibrium is the following:

$$wage(e) = \begin{cases} 13 & \text{if } e = 3 \\ 6 & \text{if } e \neq 3 \end{cases} \tag{1}$$

See the end of the answer sheet for a picture.

c. Use a new graph and a verbal explanation to demonstrate that the equilibrium you constructed in part b does not satisfy the intuitive criterion. Clearly state which part of your wage function fails the criterion.

Consider a worker who gets education $e = 9$. Any wage function which supports $(e_L = e_H = 3)$ as a pooling equilibrium must put at least some weight on a worker with $e = 9$ being a low type. If firms thought a worker with $e = 9$ were a high type, they would pay a wage of 20, and a high type would receive a payoff of 11, meaning the high type would prefer to switch from $e = 3$, where he gets utility of 10. However, low types can only be made worse off by choosing $e = 9$, no matter what the wage is. Even if a low type is paid the maximum wage of 20 (the productivity of a high type), his utility would be only $20 - 2 * 9 = 2$, whereas he gets utility 7 from choosing 3 units of education. Therefore, the intuitive criterion requires firms to hold belief $\mu(H|e = 9) = 1$, in which case they must pay a wage of 20 for any worker who chooses 9 units of education. This breaks the equilibrium identified in part b.

d. Suppose that $A = 20$ and $B = 2$. Construct a wage function so that there a separating equilibrium in which normal types get education $e_N = 0$, while smart types gets $e_S = 10$. Depict the equilibrium graphically.

One wage function that would support these education levels as a separating equilibrium is the following:

$$wage(e) = \begin{cases} 20 & \text{if } e = 10 \\ 6 & \text{if } e \neq 10 \end{cases} \tag{2}$$

See the end of the answer sheet for a picture.

e. Use a new graph and a verbal explanation to demonstrate that the equilibrium you constructed in part d does not satisfy the intuitive criterion. Clearly state which part of your wage function fails the criterion.

Consider a worker who chooses $e = 8$. Any wage function that is part of an equilibrium must pay $wage(8) < 20$, otherwise the high type of worker would surely choose to switch to $e = 8$. But the intuitive criterion says that a worker choosing $e = 8$ must be a high type, since the low type of worker could only be made worse off relative to his equilibrium payoff by choosing $e = 8$. The low type gets utility of 6 in the equilibrium of part d, yet even if he were paid the maximum wage of 20, would get utility of only 4 from choosing $e = 8$. Therefore, the intuitive criterion requires $wage(e = 8) = 20$, which breaks the equilibrium in part d.

f. Describe, using a graph and words, the unique equilibrium outcome (e_N, e_S) of this game that satisfies the intuitive criterion.

The unique equilibrium satisfying the intuitive criterion is a separating equilibrium where the high types get just enough education to leave the low types indifferent between switching to e_H and staying at $e_L = 0$. Since a choice of $e = 0$ gives a low type utility 6, a choice of $e = 7$ and a wage of 20 would give the low type the same utility. Hence, the unique equilibrium outcome is $e_L = 0, e_H = 7$, and this is supported by a wage function such as the following:

$$wage(e) = \begin{cases} 20 & \text{if } e = 7 \\ 6 & \text{if } e \neq 7 \end{cases} \tag{3}$$

g. For general values of A and B, determine the unique equilibrium outcome (e_N, e_S) satisfying the intuitive criterion.

To leave the low type indifferent between switching and not, e_H must satisfy $6 = A - be_H$, so $e_H = \frac{A-6}{b}$.

h. Explain verbally how the outcome in g is affected by an increase in A. Explain intuitively why this is the case. Do the same for an increase in B.

e_H is increasing in A and decreasing in b. The intuition is that as A increases, imitating a high type becomes relatively more valuable, and so high types must choose a higher level of education to deter the low types from choosing e_H. As b increases, education becomes more costly for the low types, and so the high types do not need to choose as high of a level of e_H to deter the low types from imitating the high types.

Problem 2 This problem asks you to consider an extension of the basic Spence model to one in which education is productive and the cost of education is convex.

Suppose that high types with education e have productivity $y(H, e) = 10 + 2e$, while low types have productivity $y(L, e) = 2 + e$. Firms cannot observe whether a worker is a high type or a low type, but know that half of all workers are of each type. A competitive labor market ensures each type of worker is paid her expected productivity. A high type can acquire e units of education at cost $c_H(e) = \frac{1}{10}e^2$, while education costs a low type $c_L(e) = \frac{1}{4}(e + 2)^2 - 1$.

a. Suppose $e_L = 0$ and $e_H = 12$. What wage function would support this outcome as a separating equilibrium? Draw a picture and/or describe using an equation.

Consider the following wage function:

$$wage(e) = \begin{cases} 34 & \text{if } e = 12 \\ 2 + e & \text{if } e \neq 12 \end{cases} \tag{4}$$

High types will maximize their utility by choosing $e = 12$ (giving utility of 19.6), while low types will maximize their utility by choosing $e = 0$ (giving utility of 2, versus -14 from deviating to $e = 12$).

b. Does the equilibrium you described in part a satisfy the intuitive criterion? Why or why not?

No. Consider education level $e = 10$. If $wage(e = 10) = 30$, high types would prefer $e = 10$ to $e = 12$, since their utility would increase to 20. Low types would not like to switch to $e = 10$ from $e = 0$ regardless of how high the wage is, since even if $wage(10) = 30$, low types utility from choosing $e = 10$ would be only -5.

c. Draw the set of all points which give the high type utility of 5. What is the slope of the indifference curve you drew, as a function of e? Determine the point of tangency between the high type's indifference curve and the function $y(H, e)$ (note that the high type may get more or less than 5 utility at the point of tangency). Do the same for the low type's indifference curve and the function $y(L, e)$.

The slope of the high type's indifference curve is $\frac{1}{5}e$, while the slope of her productivity function is 2. The point of tangency is then at $e = 10$. For the low types, the slope of the indifference curve is $\frac{1}{2}(e + 2)$, while the slope of his productivity function is 1, meaning to point of tangency is at $e = 0$.

d. Suppose that both types choose their education level so that their indifference curve is tangent to their productivity function. Describe, using a picture and/or an equation, a wage function that would support this outcome as a perfect Bayesian equilibrium.

Consider the following wage function:

$$wage(e) = \begin{cases} 30 & \text{if } e = 10 \\ 2 + e & \text{if } e \neq 10 \end{cases} \tag{5}$$

Low types strictly prefer $e = 0$ to all other education levels, and high types strictly prefer $e = 10$ to all other education levels (since they are by definition on their highest achievable indifference curve over all the points on their productivity function $10 + 2e$).

e. Does the equilibrium you described in part d satisfy the intuitive criterion? Why or why not?

Yes. Any education level other than $e = 10$ makes the high type worse off, so the intuitive criterion has no bite. (There are education levels that could potentially make only the low type better off, such as $e = 1$, but requiring $\mu(L|e = 1) = 1$ does not affect the equilibrium outcome; indeed, exactly this belief is embedded in the wage function described in part d.).

Problem 3 Consider the game in Figure 1 below.

a. Draw the reduced normal form. Find all pure strategy Nash equilibria. There is a mixed Nash equilibrium in which 1 randomizes between A and B, and 2 randomizes between L and R. Find it.

One pure Nash equilibrium, at (C, M). The mixed Nash equilibrium is at $(\frac{1}{2}A + \frac{1}{2}B, \frac{1}{2}L + \frac{1}{2}R)$.

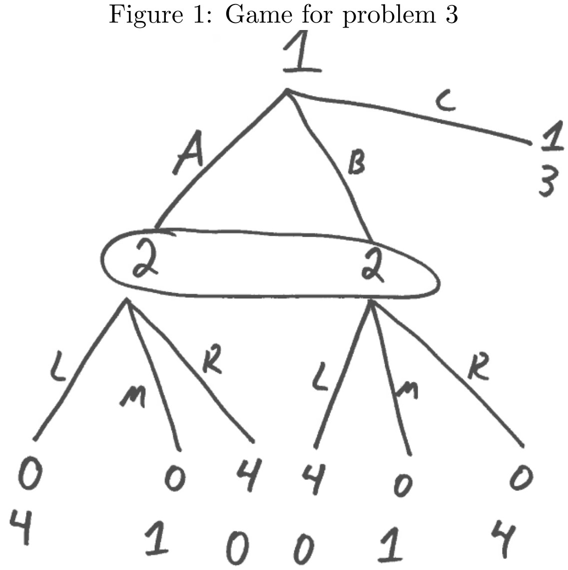
b. Find all of the game's perfect Bayesian equilibria (pure as well as mixed).

The one PBE is $(\frac{1}{2}A + \frac{1}{2}B, \frac{1}{2}L + \frac{1}{2}R)$, with 2 believing that each node is equally likely.

c. Explain in intuitive terms any differences between your answers to part a and part b.

The pure Nash equilibrium involves 2 playing a strictly dominated strategy. PBE rules this out.

Figure 1: Game for problem 3

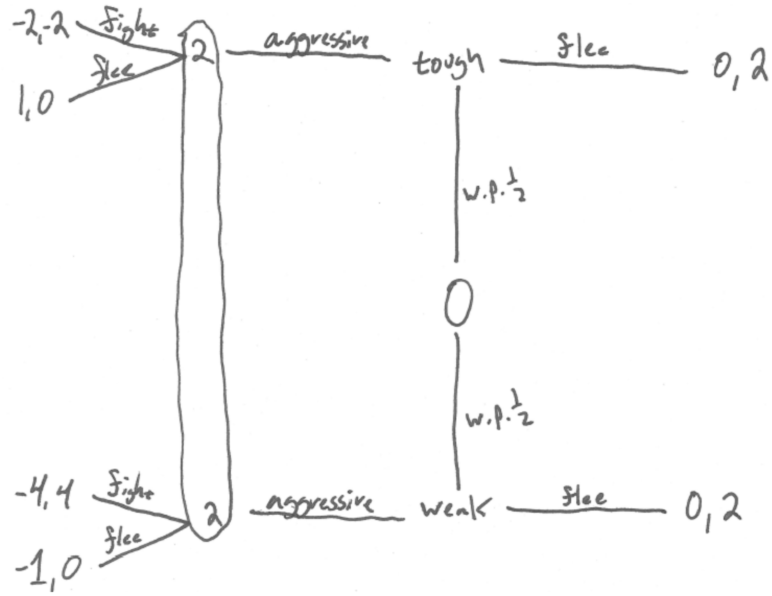


Problem 4 Consider a signaling game that satisfies:

1. Two types of player 1, tough and weak. Player 1's type is unobservable to player 2, but known to player 1. Each type is equally likely *ex ante*, and is chosen by nature.
2. Player 1 encounters player 2 in a competition for resources.
3. Either type of player 1 sends one of two signals. He can flee (game ends, player 1 receives 0, player 2 receives 2), or he can engage in costly aggressive behavior.
4. If player 1 behaves aggressively, player 2 can fight or flee. In this case, payoffs are as follows (the first number is player 1's payoff):

	fight	flee
tough	$-2, -2$	$1, 0$
weak	$-4, 4$	$-1, 0$

a. Draw the extensive form of the game described above (hint: your picture will be similar, though not identical, to the signaling games we studied in class).



b. Is there a pooling equilibrium in which both types of player 1 behave aggressively? If so, clearly state what the equilibrium is, and determine whether it satisfies the intuitive criterion. No. Regardless of which action player 2 plays, the weak type would prefer to switch to "flee'.'

c. Is there a pooling equilibrium in which both types of player 1 flee? If so, clearly state what the equilibrium is, and determine whether it satisfies the intuitive criterion. Yes. If player 2 plays "fight," neither type of player 1 wish to deviate from "flee'.' For "fight" to be optimal for player 2, she must believe she is at the lower node with probability of at least $\frac{1}{3}$. This equilibirum does not satisfy the intuitive criterion, however, as player 2's belief places positive probability on player one being a weak type conditional on player 1 behaving aggressively. But the weak player 1's payoffs from aggressive behavior are dominated by his equilibrium payoff of 0. The tough player 1, on the other hand, could potentially increase his payoff by switching to aggressive behavior. Hence, the intuitive criterion requires that player 2 believe that if player 1 were to behave aggressively, then player 1 is tough with probability 1. Under this belief, player 2 would play "flee," which breaks the equilibrium as then the tough type of player 1 actually would prefer to switch.

d. Is there a separating equilibrium in which the tough type behaves aggressively and the weak type flees? If so, clearly state what the equilibrium is, and determine whether it satisfies the intuitive criterion. Yes. Player 2 plays "flee", and neither type of player 1 wishes to deviate. The intuitive criterion is trivially satisfied, as there are no unsent signals.

Problem 5 Consider a version of the Cournot oligopoly game in which firm 2's costs are unknown to firm 1. Firm 2 knows its own cost, however. Specifically,

Inverse demand: $P = 1 - q_1 - q_2$

Firm 1's marginal cost: 0

$$\text{Firm 2's marginal cost: } \begin{cases} c_L = .2 & \text{with probability } \frac{1}{4} \\ c_H = .4 & \text{with probability } \frac{3}{4} \end{cases}$$

a. Suppose firm 2 is low cost (so that its marginal cost is $c_L = .2$). Solve for the value of q_2^L that maximizes firm 2's profits. (hint: your answer should be a function of q_1) If low cost, firm 2 solves:

$$\max_{q_2}(1 - q_1 - q_2)q_2 - .2q_2$$
$$\Rightarrow q_2^L = \frac{2}{5} - \frac{1}{2}q_1$$

b. Suppose firm 2 is high cost (so that its marginal cost is $c_L = .4$). Solve for the value of q_2^H that maximizes firm 2's profits. If high cost, firm 2 solves:

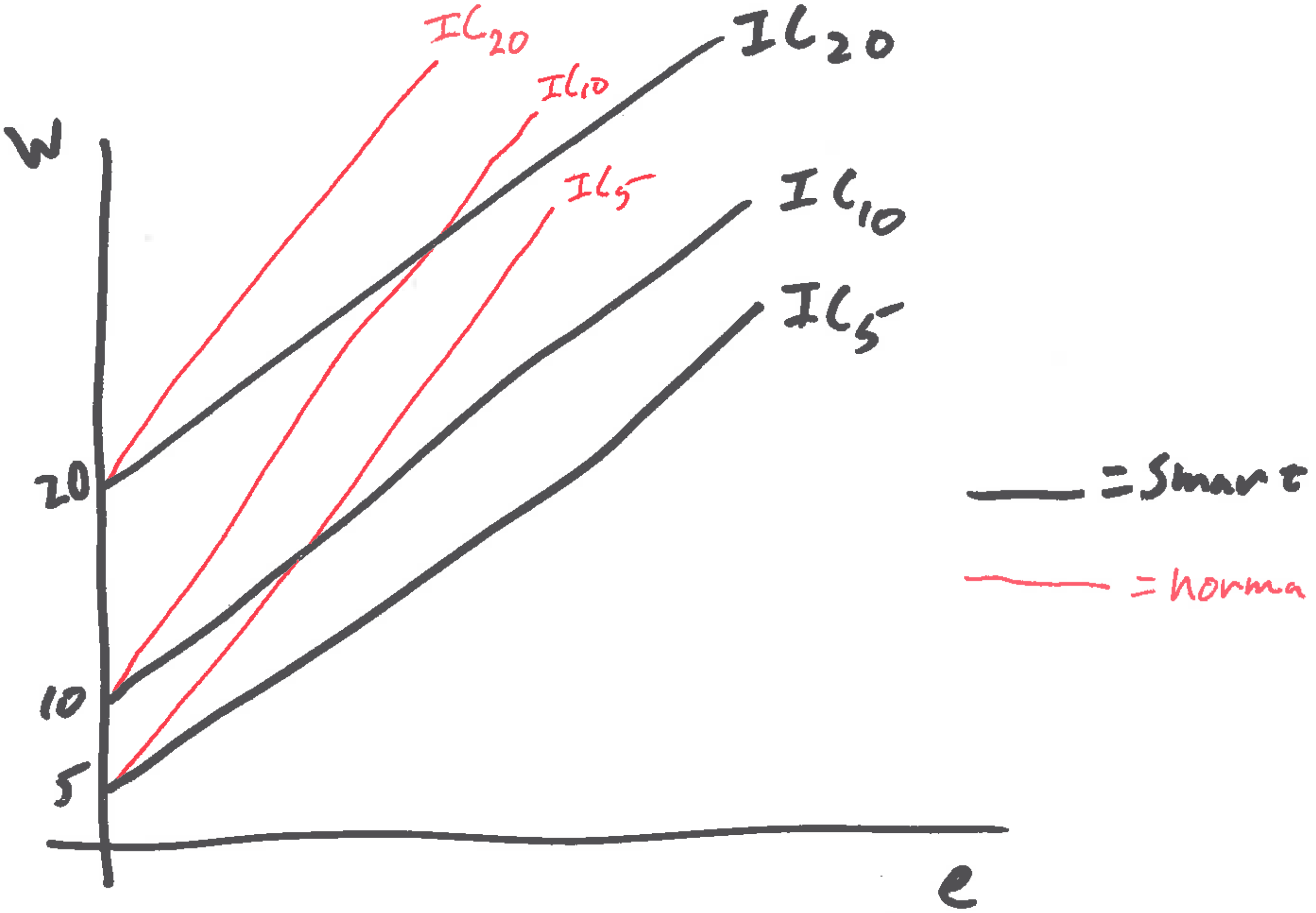
$$\max_{q_2}(1 - q_1 - q_2)q_2 - .4q_2$$
$$\Rightarrow q_2^H = \frac{3}{10} - \frac{1}{2}q_1$$

c. Solve for the value of q_1 that maximizes firm 1's profits for any values of q_2^L and q_2^H. Firm 1 solves:

$$\max_{q_1} \frac{1}{4}(1 - q_1 - q_2^L)q_1 + \frac{3}{4}(1 - q_1 - q_2^H)q_1$$
$$\Rightarrow q_1 = \frac{1}{2} - \frac{1}{8}q_2^L - \frac{3}{8}q_2^H$$

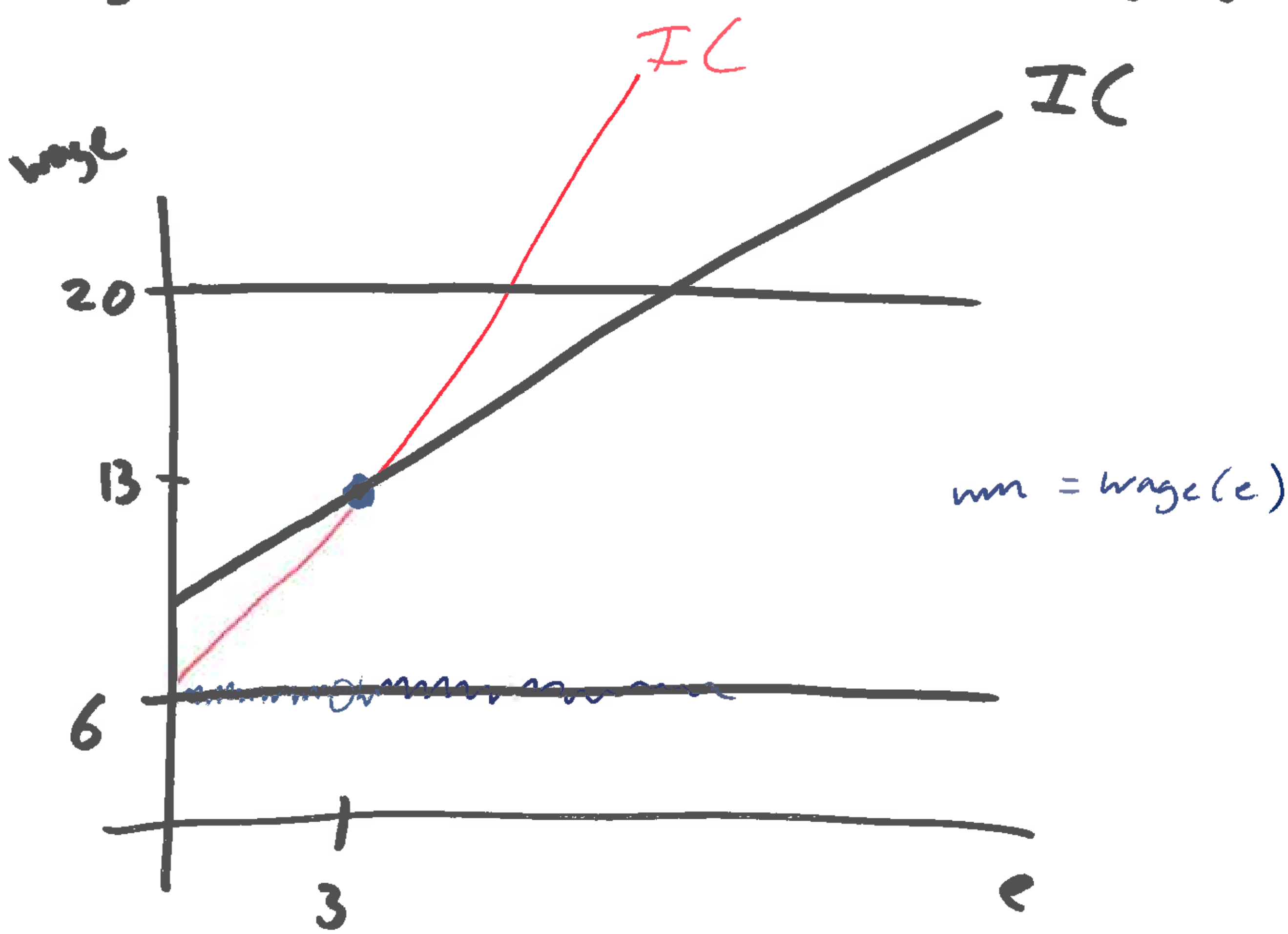
d. Clearly describe the oligopoly game's Bayesian Nash equilibrium. $q_1 = .45$, $q_2^H = .075$, $q_2^L = .175$

1a

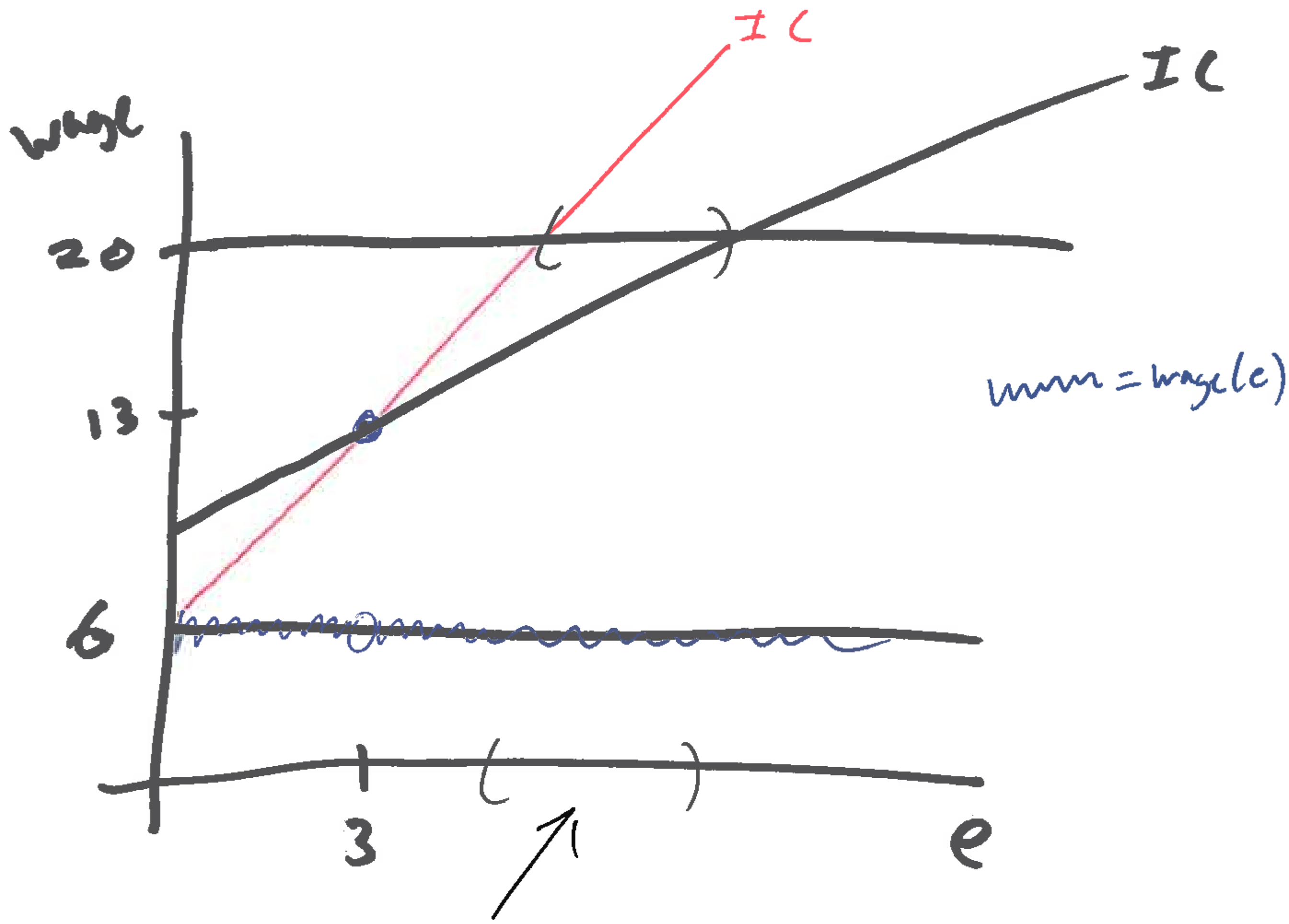


Smart workers' indifference curves

1b.

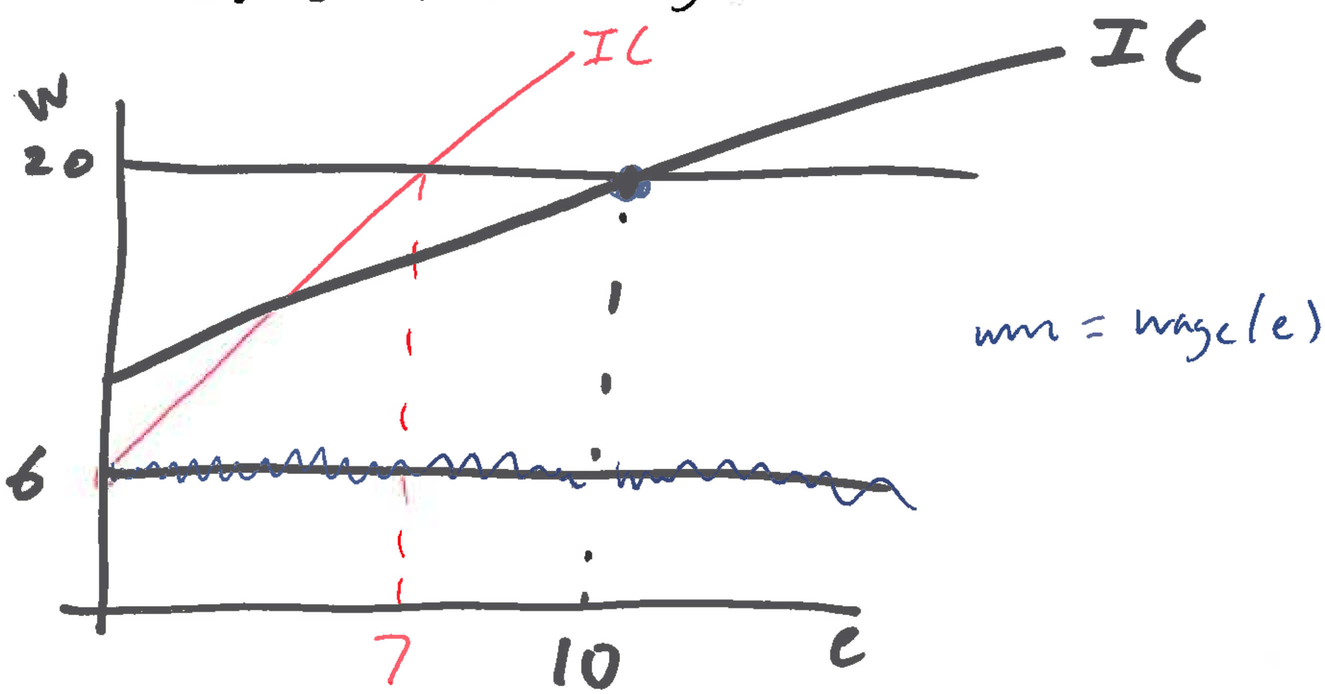


1c.

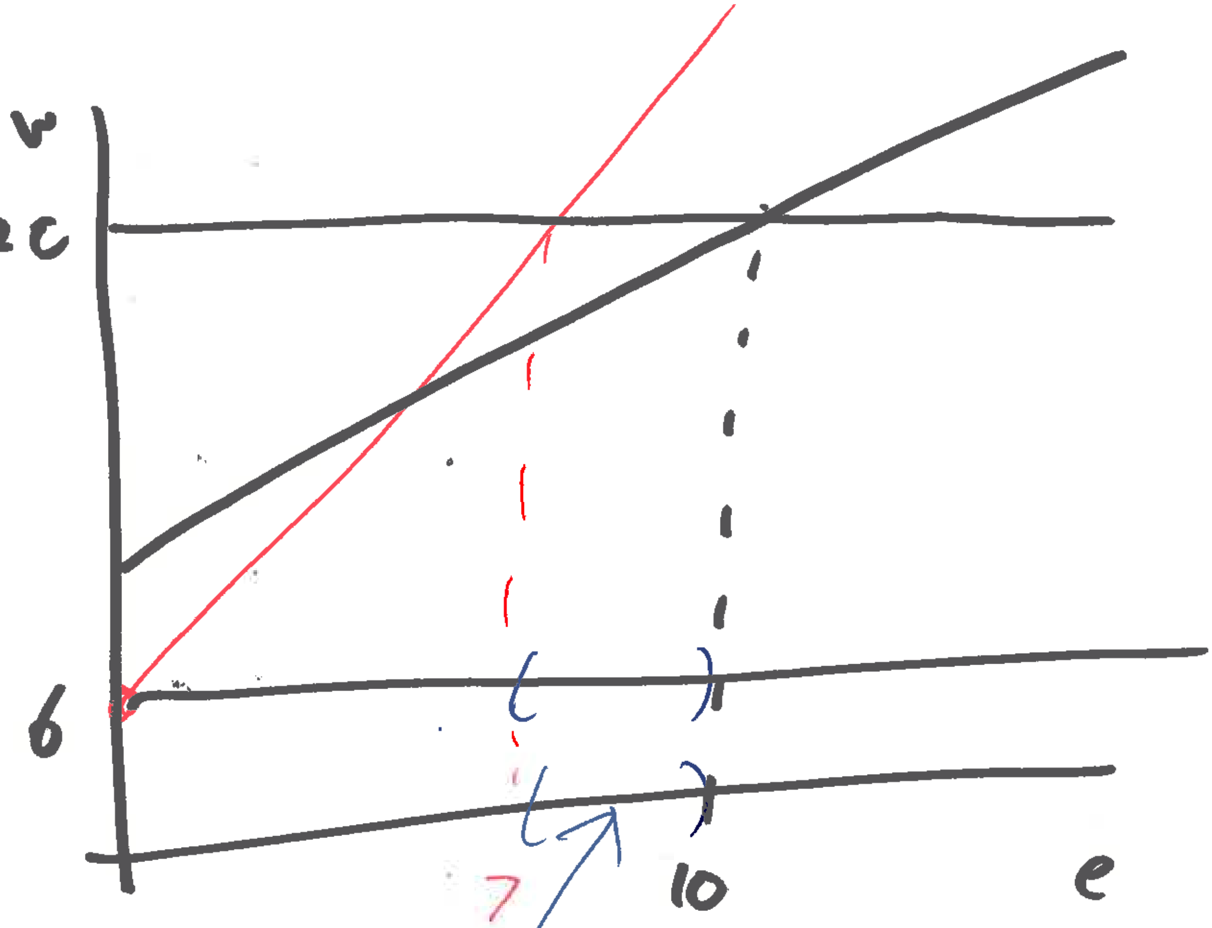


For any education level in this range, the firm should pay a wage of 20, as only the ~~hig~~ smart workers would possibly choose a value of e in this range.

1d.



1e



for any education level between 7 and 10, the firm should recognize that only the high smart type is potentially better off, and thus should pay a wage of 20.

1f.