Final exam

5/6/11

Note: Throughout, points will be deducted mercilessly for insufficiently supported answers. When in doubt, err on the side of maximum verbosity.

Problem 1 (50 points) Consider a market with two or more firms and a continuum of workers. Each firm has two types of jobs, "old" jobs and "new" jobs. The profit to the firm and the payoff to the worker, when the worker is assigned to an old job, is 0. The payoff to a worker assigned to a new job is 1. The payoff to a firm when assigning the worker to the new job is 1 if the worker is skilled, and -1 if the worker is not skilled (all payoffs already include wages). A worker must pay a cost of c to acquire skills. The value of c differs across different workers, with c being uniformly distributed on $[0, 1]$.

a. Suppose that workers first decide whether to acquire skills and then are matched to firms, who assign them to jobs. Suppose that the firms *can* observe whether each worker has acquired skills. Find the pure-strategy equilibrium job-assignment and skill-acquisition decisions.

If investment decisions are observed, firms will place anyone who has acquired skills into a new job, and anyone who has not into an old job. Given this, all workers will acquire skills.

b. Now suppose that workers first decide whether to acquire skills and then are matched to firms, who assign them to jobs. Suppose that firms *cannot* observe whether a worker has acquired skills. Find the pure-strategy equilibrium and skill-acquisition decisions.

If firms assign everyone to new jobs, no one will invest, as job placement is independent of skill acquisition. If firms assign all workers to old jobs, again, no one will invest. Therefore, the only equilibrium is for no worker to acquire skills, and for firms to place all workers into old jobs.

c. Now suppose that workers first decide whether to acquire skills, then take a test, and then are matched to firms, who assign them to jobs. Firms cannot observe whether a worker has acquired skills, but can observe the outcome of the test, which is either a pass (p) or fail (f).[1] A worker who has acquired skills passes the test with probability $\alpha > \frac{3}{4}$ and fails with probability $1 - \alpha$, while a worker who has not acquired skills passes with probability $1 - \alpha$ and fails with probability α. Find the equilibrium job-assignment and skill-acquisition decisions. (There are multiple such equilibria. Find all the pure strategy equilibria first. Consider mixed strategy equilibria if time permits.)

There are three possible firm actions in a pure strategy equilibrium: they can place all workers in new jobs, place all workers in old jobs, or place only workers who pass the test into new jobs. If they place all workers into new jobs, then no worker will acquire skills, and so the firm would clearly be better off placing workers into old jobs, and this is not an equilibrium.

If firms place all workers into old jobs, no worker wants to acquire skills, and so firms are justified in putting all workers into old jobs (and firms believe that fraction $\pi = 0$ of workers are qualified, and continue to believe this after either test result).

[1] This is a similar setup to a model studied in class, but note that here the test has only two possible outcomes, whereas in class, the test score was continuously measured.

If firms promote only workers who pass the test, then a worker will acquire skills if:

$$\begin{aligned} \alpha - c &\geq 1 - \alpha \\ c &\leq 2\alpha - 1 \end{aligned} \tag{1}$$

and so fraction $2\alpha - 1$ of all workers acquire skills under this scenario. To check to see if this is an equilibrium, note that in any equilibrium a firm's belief prior to seeing the test result that a given worker is qualified is $2\alpha - 1$. If the worker passes the test, this updates to a posterior of:

$$\begin{aligned} \mu_p &= \frac{(2\alpha - 1)\alpha}{(2\alpha - 1)\alpha + 2(1-\alpha)^2} \\ &= \frac{2\alpha^2 - \alpha}{4\alpha^2 - 5\alpha + 2} \end{aligned} \tag{2}$$

In order for the firm's strategy to be optimal, it must be that $\mu_p \geq \frac{1}{2}$ and $\mu_f \leq \frac{1}{2}$. Checking the first condition from (2) gives that $\mu_p \geq \frac{1}{2}$ iff $\alpha \geq \frac{2}{3}$. An analogous condition gives that $\mu_f \leq \frac{1}{2}$ for all $\alpha \geq \frac{3}{4}$, and so the firms are optimizing given worker behavior. This is an equilibrium. I will not take the time to solve for mixed strategy equilibria here, but this would be a good exercise to prepare for prelims.

d. Now suppose that workers come in two varieties, red and green. The colors have no effect on the cost of acquiring skills, test outcomes, the value of acquiring skills, or anything else, but are observed by firms. Is there an equilibrium in which different colored workers behave differently?

Yes. There are two equilibria in part c, one in which firms have a prior of 0 and never promote anyone, and one in which firms have a prior of $2\alpha - 1$ and promote all workers who pass the test. Apply one equilibrium to red workers, the other to green workers, and we have a discriminatory equilibrium, even though red and green workers are *ex ante* identical.

Problem 2 (15 points) Consider a variation on the education signaling game in which, first, nature draws a worker's productivity from some continuous distribution on $[\underline{\theta}, \overline{\theta}]$. Once the worker observes her type, she can choose whether to submit to a costless test that perfectly reveals her ability. Finally, after observing whether the worker has taken the test and its outcome if she has, two firms bid for the worker's services. Argue that in any subgame perfect equilibrium of this model all worker types submit to the test, and firms offer a wage no greater than $\underline{\theta}$ to any worker not doing so. (hint: note that there are two major differences from the education model studied in class. 1-There is no education, 2-there is a continuum of worker types.)

Suppose there is an equilibrium in which workers in some set $X \subset [\underline{\theta}, \overline{\theta}]$ do not submit to the test. Then, firms will pay all workers in this group a wage equal to $E[\theta | \theta \in X]$. But then consider the worker with productivity equal to $\max W$; if he submits to the test, he will be paid a wage equal to $\max W > E[\theta | \theta \in X]$, and so is better off taking the test. As this is a contradiction, conclude that there is no equilibrium with some workers not taking the test. (note: some minor measure-theoretic issues are ignored in this answer).

Problem 3 (35 points) Consider an oligopoly with K firms. Each firm can produce costlessly, and market demand is given by $P = 1 - \sum_{i=1}^{K} q_i$.

a. Suppose $K = 3$, and the three firms are Cournot competitors. Solve for the Nash equilibrium profits of firms 1, 2, and 3 in this game.

Each firm will produce a quantity of $q = \frac{1}{4}$, the market price will be $P = \frac{1}{4}$, and each firm will earn profit of $\frac{1}{16}$.

b. Suppose $K = 3$. Suppose that firm 1 publicly commits to a quantity. After observing firm 1's choice of quantity, firms 2 and 3 simultaneously choose quantities. Solve for the subgame perfect equilibrium profits of firms 1, 2, and 3 in this game.

Firms 2 and 3 will each produce quantity $\frac{1-q_1}{3}$. Given this, firm 1 chooses q_1 to solve the following:

$$\max_{q_1}[1 - q_1 - \frac{2}{3}(1 - q_1)]q_1 \tag{3}$$

which has solution $q_1 = \frac{1}{2}$, meaning that firms 2 and 3 each produce quantity $\frac{1}{6}$, the market price is $\frac{1}{6}$, firm 1's profit is $\frac{1}{12}$, and firms 2 and 3 each earn profit of $\frac{1}{36}$.

For parts c-e: Suppose that K firms choose quantity sequentially (that is, firm 1 publicly announces a quantity. After observing q_1, firm 2 publicly announces a quantity. After observing q_1 and q_2, firm 3 publicly announces a quantity, and so on).

c. Suppose $K = 3$. Solve for each firm's profit in the subgame perfect equilibrium of this game. You are not allowed to directly reference the result of parts d and e.

Firm 3 will produce $\frac{1}{2} - \frac{1}{2}(q_1 + q_2)$. Given this, 2 chooses q_2 to solve:

$$\max_{q_2}(1 - q_1 - q_2 - (\frac{1}{2} - \frac{1}{2}(q_1 + q_2))q_2$$

which has solution $q_2 = \frac{1}{2} - \frac{1}{2}q_1$. Finally, given this, 1 chooses q_1 to solve:

$$\max_{q_1}[1 - q_1 - (\frac{1}{2} - \frac{1}{2}q_1) - (\frac{1}{4} - \frac{1}{4}q_1)]q_1 \tag{4}$$

which has solution $q_1 = \frac{1}{2}$. Therefore, $q_2 = \frac{1}{4}$ and $q_3 = \frac{1}{8}$. Market price is $\frac{1}{8}$, and profits are $(\pi_1, \pi_2, \pi_3) = (\frac{1}{16}, \frac{1}{32}, \frac{1}{64})$.

d. Now suppose K is a large number. Solve for the SPE of the game via guess and verify (I guess, you verify).

$$\textbf{Guess: } q_k = \frac{1 - \sum_{i=1}^{k-1} q_i}{2} \tag{5}$$

Verify this guess[2] by showing that firm 1's best response to firms $2, 3, ..., K$ playing this strategy is $q_1 = \frac{1}{2}$. Do this in two steps:[3]

[2]Think about what this guess is saying. Each firm produces enough quantity to satisfy half the remaining demand on its turn.

[3]There may be a third step required to verify the guess. Do not worry about this. If you cannot do step 1, you may take its result as given and show step 2 alone for partial credit.

1. Show that if firms $2, 3, ..., K$ produce according to (5), then $q_k(q_1) = \frac{1}{2^{k-1}} - \frac{1}{2^{k-1}}q_1$.
2. Show that 1 implies $q_1^* = \frac{1}{2}$.

For step 1- the guess defines q_k recursively:

$$\begin{aligned}
q_2 &= \frac{1}{2} - \frac{1}{2}q_1 \\
q_3 &= \frac{1}{2} - \frac{q_1}{2} - \frac{\frac{1}{2} - \frac{q_1}{2}}{2} = \frac{1}{4} - \frac{1}{4}q_1 \\
q_4 &= \frac{1}{2} - \frac{q_1}{2} - \frac{\frac{1}{4} - \frac{1}{4}q_1}{2} = \frac{1}{8} - \frac{1}{8}q_1 \\
q_5 &= \frac{1}{2} - \frac{q_1}{2} - \frac{\frac{1}{4} - \frac{1}{4}q_1}{2} - \frac{\frac{1}{8} - \frac{1}{8}q_1}{2} = \frac{1}{16} - \frac{1}{16}q_1 \\
&... \\
q_k &= \frac{1}{2^{k-1}} - \frac{1}{2^{k-1}}q_1
\end{aligned}$$

Looking for a pattern like this is fine; you can also do this more formally by careful substitutions. For step 2- Firm 1 then chooses q_1 to solve:

$$\begin{aligned}
&\max_{q_1}(1 - q_1 - \sum_{k=2}^{K} q_k)q_1 \\
\text{simplifying, } &\max_{q_1}(1 - q_1 - \sum_{k=1}^{K-1}\left(\frac{1}{2}\right)^k (1 - q_1))q_1 \\
\text{simplifying further, } &\max_{q_1}(1 - q_1 - (1 - \frac{1}{2}^{K-1})(1 - q_1))q_1 \\
\text{simplifying still further, } &\max_{q_1} \frac{1}{2}^{K-1}(1 - q_1)q_1
\end{aligned}$$

which has solution $q_1 = \frac{1}{2}$, as was to be shown. (note: it does not immediately follow from this that firm 2 is optimizing by playing $\frac{1}{2} - \frac{1}{2}q_1$. The best way to show this would be to use a similar proof to show that if demand is $a - \sum_{k=1}^{K} q_k$ then firm 1 optimally plays $\frac{a}{2}$, and then it does follow that $q_2 = \frac{1}{2} - \frac{1}{2}q_1$. It would be a good exercise to make sure you understand the distinction.)

e. Calculate firm k's SPE profits in the game described in part d. (Whether or not you were able to solve part d, you may take its result as given).

The guess above is that $q_1 = \frac{1}{2}$, $q_2 = \frac{1}{4}$, ... $q_k = \frac{1}{2^k}$. Therefore, market price is given by:

$$P = 1 - \sum_{k=1}^{K}\left(\frac{1}{2}\right)^k = \frac{1}{2}^K \tag{6}$$

and so profit is given by:

$$\pi_k = \frac{1}{2}^K * \frac{1}{2}^k = \frac{1}{2}^{K+k}$$