Homework 3

answers

Problem 1 Mas-Colell, problem 9.B.7

Suppose T is even. Then 1 makes an offer in the final period. Obviously, he'll offer $(v, 0)$, which will be accepted. Then, in period $T-1$, 2 will offer $(v-c, c)$, which will be accepted. Then, in $T-3$, 1 will offer $(v, 0)$, and 2 will accept. From here, it is clear that 1's intial offer in period 0 will be $(v, 0)$, and it will be accepted (technically, this is a proof by mathematical induction).

If T is odd, then backward induction tells us that 1's initial offer will be $(c, v-c)$, and it will be accepted. Thus who is better off in this modified game depends entirely on who makes the offer in the final period. The person making the final offer gets a payoff of $v-c$, his opponent a payoff of 0. As $T \to \infty$, no compelling pattern emerges (mathematically, the limit is undefined). The erratic behavior of this alternative model suggests the standard setup where players discount future payoffs and offers are made costlessly may be a better option for modeling this game.

Problem 2 Mas-Colell, problem 9.B.9

The normal form game has 2 pure strategy Nash equilibria, (a_2, b_2) and (a_3, b_3), with payoffs $(5, 5)$ and $(1, 1)$, respectively. Therefore, in any SPE of the repeated game, each player will get a payoff of either 5 or 1 in the second round. The fact that there are two equilibria suggests a "punishment" technique, which they can use to ensure higher payoffs. The players commit to playing the $(5, 5)$ equilibrium in period 2 if both players go along with the prescribed behavior in period 1, and the $(1, 1)$ equilibrium is played otherwise. Since the payoff difference is 4, any period 1 outcome in which neither player can profit by more than 4 by switching strategies can be supported in a SPE. For example, suppose the players agree to play (a_1, b_1) in period 1, and play (a_2, b_2) in period 2 if (a_1, b_1) is played in period 1, and (a_3, b_3) in period 2 if anything else is played. These strategies represent a SPE in which both players get a payoff of 15. Thus, every strategy profile $s \in S \backslash \{(a_2, b_3), (a_3, b_2)\}$ can be supported in the first round with the punsihment technique I suggest. There are also SPE in which one of the stage game NE is played in each of the two periods.

Problem 3 The "modified centipede" we considered in class (see the non-answer key version of this homework) has a sequential equilibrium in which both players continue. In finding this equilibrium, we said that, from Bayes' rule, $\mu_1(y) = \frac{19\sigma_2(c^1)}{19\sigma_2(c^1)+1}$ if $\sigma_1(C^1) > 0$. Since all information sets are reached with positive probability if $\sigma_1(C^1) > 0$, the belief $\mu_1(y)$ is trivially consistent, and so any equilibrium involving $\sigma_1(C^1) > 0$ will be a sequential equilibrium.

Now, argue that *even if* $\sigma_1(C^1) = 0$, the belief $\mu_1(y) = \frac{19\sigma_2(c^1)}{19\sigma_2(c^1)+1}$ is still the only consistent belief. Do this in two steps:

a. Show that if $\sigma_1(C^1) = 0$ and $\sigma_2(c^1) > 0$, for any sequences $\epsilon^k \to 0$, $\alpha^k \to \sigma_2(c^1)$, the limit of the beliefs $\mu^k(y)$ implied by strategies $\sigma_1^k(C^1) = \epsilon^k$ and $\sigma_2^k(c^1) = \alpha^k$ is $\frac{19\sigma_2(c^1)}{19\sigma_2(c^1)+1}$

Take arbitrary sequences $\epsilon^k \to 0$ and $\alpha^k \to \sigma_2(c^1)$. Then, by Bayes rule,

$$\begin{aligned}
\mu^k(y) &= \frac{\frac{19}{20}\epsilon^k\alpha^k}{\frac{19}{20}\epsilon^k\alpha^k + \frac{1}{20}\epsilon^k} \\
&= \frac{19\alpha^k}{19\alpha^k + 1} \\
&\to \frac{19\sigma_2(c^1)}{19\sigma_2(c^1) + 1}
\end{aligned}$$

Since the sequences ϵ^k and α^k were chosen arbitrarily, this shows that for *any* such sequences, $\mu^k \to \frac{19\sigma_2(c^1)}{19\sigma_2(c^1)+1}$.

b. Show that if $\sigma_1(C^1) = 0$ and $\sigma_2(c^1) = 0$, an implication of consistency implies that $\mu_2(y) = 0$. This follows immediately from parsimony. If $\sigma_2(c^1) = 0$, then two deviations are required to reach node y (1 switches to C^1 and 2 switches to c^1, whereas only one is required to get to node z (1 switches to C^1. Therefore, parsimony implies that the only consistent belief in this case is $\mu(y) = 0 = \frac{19\sigma_2(c^1)}{19\sigma_2(c^1)+1}$.

Problem 4 It follows from problem 3 that the equilibrium in the "modified centipede" game we solved for in class in which both players continue is the unique sequential equilibrium. Does this game possess any other perfect Bayesian equilibrium profiles?

The difference between a PBE and a SE is that beliefs are required to be consistent in a SE, and are unrestricted at unreached information sets in a PBE. Notably, this means that should 1 play S^1, any belief $\mu(y)$ is allowed for 1. Suppose 1 always plays S^2 (optimal if $\mu(y) \geq \frac{4}{5}$). Then, a normal player 2 will always play s^1, and at his first info set, gets 0 from playing S^1. versus $\frac{19}{20}(-1) + \frac{1}{20} * 8 = -\frac{11}{20}$ from switching to C^1 and C^2. Thus neither player wants to switch, and this is an equilibrium (formally, the equilibrium is $((S^1, S^2), (s^1, s^2)), \mu_1(w) = \frac{19}{20}, \mu_1(y) \geq \frac{4}{5}))$.

Note why this is not a sequential equilibrium: the belief $\mu_1(y) \geq \frac{4}{5}$ is not consistent given strategies S^1 and s^1. Concistency requires 2 believe that he is certainly at node z should his information set be reached. He is, however, allowed to believe he is almost surely at node y in a perfect Bayesian equilibrium.

Problem 5 Ace-King-Queen poker is a two-card game that is played using a deck consisting of three cards: an ace (the high card), a king (the middle card), and a queen (the low card). Play proceeds as follows:

- Each player puts \$1 in a pot in the center of the table.
- The deck is shuffled, and each player is dealt one card. Each player sees only the card he is dealt.
- Player 1 chooses to raise (R) or fold (F). A choice of R means that player 1 puts an additional \$1 in the pot. Choosing F means that player 1 ends the game, allowing player 2 to have the money already in the pot.
- If player 1 raises, then player 2 chooses to call (c) or fold (f). A choice of c means that player 2 also puts an additional \$1 in the pot; in this case, the players reveal their cards and the player with the higher card wins the money in the pot.

a. Draw the extensive form of this game.

See the figure on the final page of this file. Since it is such a large game, I have not attempted to use different letters for different behavior strategies, and I have omitted the probabilities nature assigns to each type profile ($\frac{1}{6}$, in every case).

b. Find all sequential equilibria of this game.

Start with the easy parts: 2 folds if she has a queen, and calls when she has an ace. 1 raises when he has an ace. These are all dominant strategies

Given the above, 1 will raise when he has a king. As he regards it as equally likely that 2 has a queen or an ace should he have a king, his payoff to raising is $-2 * .5 + 1 * .5 = -.5$. As his payoff to folding is -1, he prefers raising to folding.

All that remains is to determine what 1 should do if he has a queen, and what 2 should do if she has a king. As usual, the way to do this is to consider all possible supports. Let's try looking at each possible move for player 2:

- 2 calls with a king. If this is the case, then 1 will always fold a queen. But in this case, 2 should believe that 1 has an ace when she has a king and her information set is reached, in which case folding is optimal. Therefore, there is no equilibrium in which 2 always calls when she has a king.
- 2 folds with a king. If this is the case, 1 will always raise with a queen, in which case 2 should think there is a 50/50 chance 1 has either an ace or a queen when she has a king, but in this case, 2 prefers to call with a king. Therefore, there is no equilibrium in which 2 always folds a king.
- 2 mixes when she has a king. For 2 to be willing to mix, her expected payoff to folding and calling must be equal. This, in turn, requires $\mu_2(A|k) = \frac{3}{4}$. For this to be a Bayesian belief, it must be that $\sigma_1(R|Q) = \frac{1}{3}$. For this to be optimal for player 1, he must be indifferent between R and F when he has a queen. This is true if $\sigma_2(c|k) = \frac{1}{3}$.

Since we did an exhaustive search for equilibria, we can conclude there is only one equilibrium, in which 1 raises with an ace, raises a king, and raises a queen $\frac{1}{3}$ of the time, and in which 2 calls with an ace, calls with a king $\frac{1}{3}$ of the time, and folds a queen. 1's beliefs are 50/50 at each of his info sets. 2 has beliefs $\mu_2(K|a) = \frac{3}{4}$, $\mu_2(A|k) = \frac{3}{4}$, and $\mu_2(A|q) = .5$.

Note that we had no need of the consistency requirement here. All perfect Bayesian equilibria are also sequential. This is because there is no PBE that has unreached information sets.

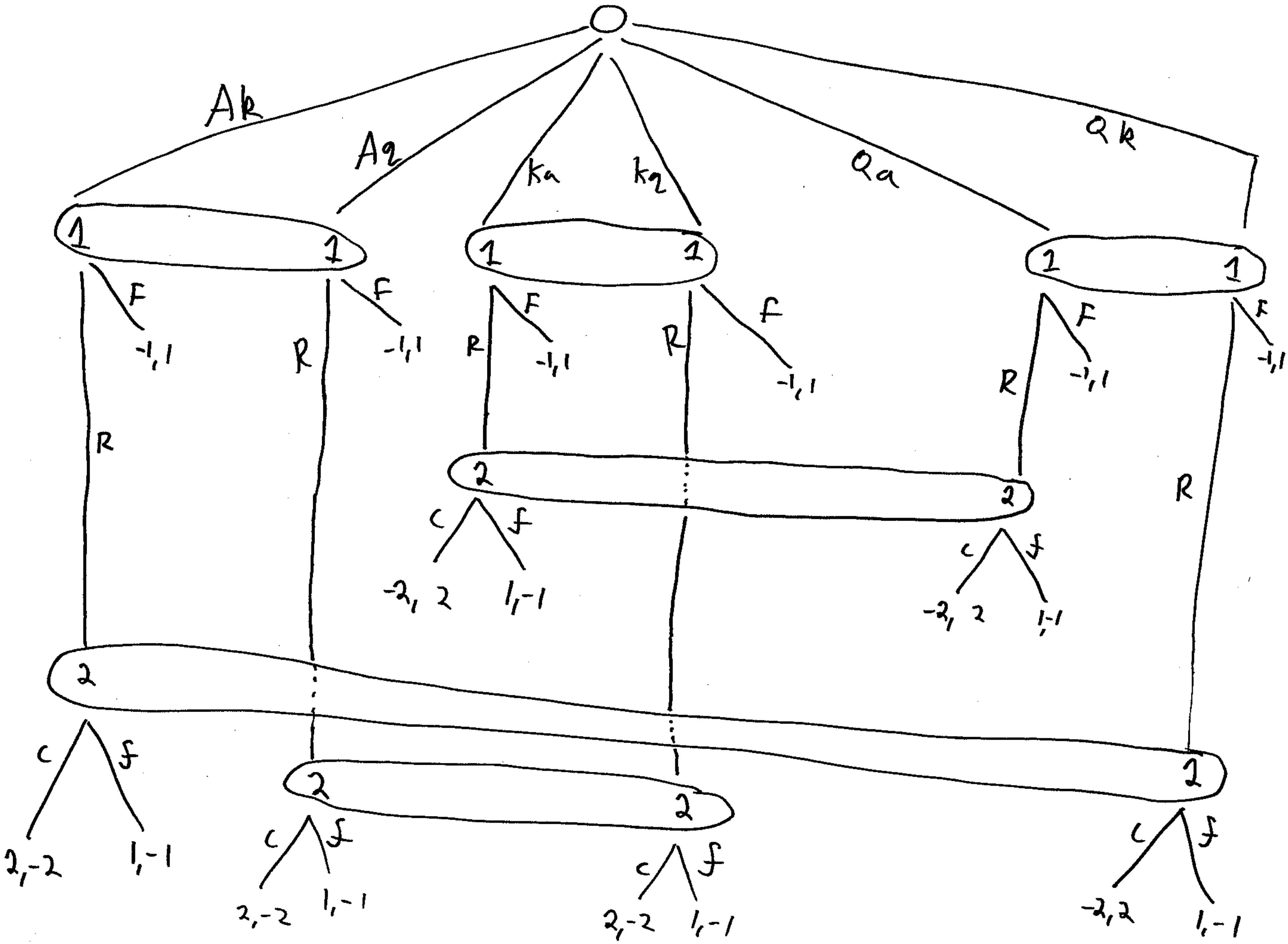
c. If you could choose to be either player 1 or player 2 in this game, which player would you choose to be?

Each type profile (Ak, Aq, Ka, Kq, Qa, Qk) is equally likely. 1's expected payoff is then

$$\frac{1}{6}\left((\frac{1}{3}*2+\frac{2}{3}*1)+1+(-2)+1+(\frac{1}{3}*(-2)+\frac{2}{3}*(-1))+(\frac{2}{3}*(-1)+\frac{2}{9}*1+\frac{1}{9}*(-2)\right) = -\frac{1}{9}$$

2's expected payoff is $\frac{1}{9}$ (you can compute this directly, or note that this is a zero-sum game, and so necessarily 2's payoff is the negative of 1's). Therefore, you are better off being the second mover in this game.

d. (optional, do not turn in) Suppose we modify the game as follows: instead of choosing between raise and fold, player 1 chooses between raise (R) and laydown (L). A choice of L means that the game ends, the players show their cards, and the player with the higher card wins the pot. Answer parts b and c for this modified game.



Ak
Aq
Ka
Kq
Qa
Qk
1
F
R
-1,1
2
c
f
2,-2
1,-1
-2,2