Perfect Bayesian equilibria

- For an extensive form game Γ, recall that H_i denotes the set of all of player i's information sets, and that $h \in H_1$ represents one particular information set.
- Player i's *beliefs* are a function $\mu_i : h \to [0,1]$ satisfying $\sum_{x \in h} \mu_i(x) = 1$ for all $h \in H_i$.
- So for all $x \in h$, $\mu_i(x)$ represents the probability i assigns to node x being reached, conditional on information set h being reached.
- Let $\mu = (\mu_1, \mu_2, ..., \mu_n)$ describe all players' beliefs together.
- Given that strategy profile σ is played, let $P_\sigma(x)$ denote the probability of actually reaching node x, and $P_\sigma(h) = \sum_{x \in h} P_\sigma(x)$ denote the probability information set h is reached.
- **Definition:** beliefs μ are *Bayesian* given strategy profile σ if:

$$\mu_i(x) = \frac{P_\sigma(x)}{P_\sigma(h)} \quad \text{whenever } P_\sigma(h) > 0$$

- Then, say that strategy i is rational for player i given μ_i σ_{-i} if at each of his information sets the following holds:

$$\sum_{x \in h} \mu_i(x) u_i(\sigma_i, \sigma_{-i}|x) \geq \sum_{x \in h} \mu_i(x) u_i(\sigma_i', \sigma_{-i}|x) \quad \text{for all } \sigma_i' \tag{1}$$

- Though (1) looks complicated, it amounts to the usual condition that a strategy is rational only if there is no alternate strategy that would yield a higher payoff, holding fixed what everyone else is doing (and, here, holding fixed a set of beliefs).
- Say strategy profile σ is *sequentially rational* given beliefs μ if, for all players and all information sets, σ_i is rational.

Definition: A *perfect Bayesian equilibrium* is a strategy-belief pair (σ, μ) satisfying:

1. μ is Bayesian given σ (i.e. comes from Bayes' rule whenever possible)
2. σ is sequentially rational given μ

While we have not thought of beliefs as essential to defining Nash equilibria, we can characterize what would be true of any beliefs in a Nash equilibrium. Contrast the definition of a PBE with that of a NE:

Definition: A *Nash equilibrium* is a strategy-belief pair (σ, μ) satisfying:

1. μ is Bayesian given σ (i.e. comes from Bayes' rule whenever possible)
2. σ is rational *at each information set on the equilibrium path induced by* σ, given σ_{-i} and μ