Final exam

answers

Instructions: Throughout, points will be deducted for insufficiently supported answers. You may use books, notes, and calculators, but no other electronic devices. You may not discuss the exam with anyone other than me until all students have turned in their exams.

Problem 1 (30 points) Consider a market with two or more firms and a continuum of workers. Each firm has two types of jobs, "good" jobs and "bad" jobs. Workers enjoy good jobs more than bad jobs; if a worker works a good job, his utility is $X \in [0, 2]$, while if he works a bad job, his utility is 0. Prior to entering the labor force, a worker chooses to either become skilled or to remain unskilled. A worker must pay a cost of c to acquire skills. The value of c differs across different workers, with c being uniformly distributed on $[0, 1]$. A firm gets a profit of 0 from any worker placed into a bad job, and a profit of 1 from a skilled worker placed into a good job and -1 from an unskilled worker placed into a good job. All payoffs include wages.

Firms cannot observe whether a worker is skilled or not, but can observe the outcome of a test that each worker takes. Specifically, a skilled worker *passes* the test with probability $\frac{3}{4}$ and *fails* with probability $\frac{1}{4}$, while an unskilled worker passes with probability $\frac{1}{4}$ and fails with probability $\frac{3}{4}$ (Important: Note that there are only two possible test scores, pass and fail, and so this setup is different than the setup we studied in class where firms chose what constituted a passing score).

a. Show that for *any* $X \in [0, 2]$ there is an equilibrium in which firms put all workers into the bad job, regardless of test score. Support your answer.

If firms put all workers into the bad job, regardless of test score, workers have no incentive to invest, and so every worker will choose not to invest. Consequently, the firm's posterior belief should be that, regardless of test score, there is a 100% chance that a worker is unqualified, and so the firm is maximizing profit by putting all workers into the bad job. As this reasoning is independent of the value of X, the claim holds.

b. Show that for *no* $X \in [0, 2]$ is there an equilibrium in which firms put all workers into good jobs, regardless of test score. Support your answer.

If firms were to put all workers into the good job, independently of test score, then no worker would choose to invest in becoming qualified. Given this, a firm should have the posterior that a worker is 100% likely to be unqualified, regardless of test score, and so firms would put workers into bad jobs to maximize profits.

c. For what values of X is there an equilibrium in which firms put all workers who pass the test into good jobs and put all workers who fail the test into bad jobs? Support your answer. (Hint: a key step to answering this part is calculating what the firm's prior and posterior beliefs are, where posterior beliefs depend on whether or not a worker passed. In equilibrium, these posteriors need to be consistent with firm profit maximization).

If firms put workers who pass into good jobs, and those who fail into bad jobs, then a worker invests in becoming qualified iff

$$c \leq \frac{3}{4}X - \frac{1}{4}X = \frac{1}{2}X \tag{1}$$

Given that $c \sim U[0, 1]$, fraction $\frac{1}{2}X$ of workers become qualified. Therefore, a firm's prior belief that a worker is qualified is $\pi = \frac{1}{2}X$. The posterior that a worker who passes the test is qualified is then:

$$p(\text{qualified}|\text{pass}) = \frac{\frac{1}{2}X\frac{3}{4}}{\frac{1}{2}X\frac{3}{4} + (1-\frac{1}{2}x)\frac{1}{4}} = \frac{\frac{3}{2}X}{X+1} \tag{2}$$

For putting such a worker into a good job to be consistent with firm profit maximization, we need $p * 1 + (1-p) * -1 \geq 0$, or $p \geq \frac{1}{2}$. From inspection of (2) above, we get that this is the case iff $X \geq \frac{1}{2}$.

Finally, it must also be the case that a firm prefers to put a failing worker into the bad job. A firm's posterior that a worker is qualified given that he has failed the test is:

$$p(\text{qualified}|\text{fail}) = \frac{\frac{1}{2}X * \frac{1}{4}}{\frac{1}{2}X * \frac{1}{4} + (1-\frac{1}{2}X)\frac{3}{4}} = \frac{\frac{1}{2}X}{3-X} \tag{3}$$

For putting such a worker into a bad job to be optimal, it must be that $p \leq \frac{1}{2}$, or $X \leq \frac{3}{2}$.

Conclude that there is an equilibrium in which firms put passing workers into good jobs and failing workers into bad jobs iff $X \in [\frac{1}{2}, \frac{3}{2}]$.

Problem 2 (15 points) True/false/uncertain: In the Spence education signaling model studied in class, the separating equilibrium satisfying the intuitive criterion is Pareto superior to the pooling equilibrium (in which all workers get zero education and are all paid the same wage). Support your answer.

This is false. Consider the example pictured on the final page of this answer set. Low types are better off than they would be in the separating equilibrium satisfying the intuitive criterion, but high types are worse off. The reason is because high types need to get costly education to separate from the low types, while they would be better off taking a lower wage and zero education in the pooling equilibrium.

Problem 3 (25 points) Little Airlines' Lexington-Mumbai route is flown by both tourists and business travelers. Tourists (80% of all travelers) have demand $p = 30 - q$ for quality level q, while business travelers (20% of all travelers) have demand $p = 40 - q$. For the sake of simplicity, assume that it does not cost the airline anything to change the quality levels in its plane, and that capacity is not a concern; the plane used on this route is big enough to hold all travelers.

a. Currently, Little Airlines has 2 sections on its plane: coach, with quality 30, and business class, with quality 40. What prices should it set for a coach ticket and a business class ticket so that tourists buy coach tickets and business travelers buy business class tickets, and so that Little Airlines maximizes profits?

Little Airlines should charge a price of $400 for a coach ticket, and $500 for a business class ticket.

b. Explain intuitively why the quality levels identified in part a. are not profit-maximizing for Little Airlines.

If Little Airlines were to slightly lower the quality in coach, the price of a coach ticket would decrease only slightly, whereas the price they could charge for a business class ticket would go up by more than slightly.

c. Suppose the fraction of travelers flying this route on business were to increase. Explain intuitively what effect, if any, this would have on the optimal price of a coach ticket.

Little Airlines will optimally lower the quality in coach until the marginal benefit to doing so (a higher business class price) equals the marginal cost of doing so (a lower coach ticket price). When there are more business travelers, the airline will be relatively more concerned about the business class price, and so will be more willing to lower the quality in coach.

d. Determine the profit-maximizing prices and quality levels for both coach and business class, both when 80% of all travelers are tourists, and when 90% of all travelers are business travelers. How does the fraction of business travelers affect the quality in coach?

Following the solution technique from class, let A be the price of a coach ticket, and $A+C$ be the price of a business class ticket. It is trivial that the quality in business class will be 40. Write A and C as functions of q, the coach quality:

$$A = 450 - \frac{1}{2}(30-q)^2$$
$$C = \frac{1}{2}(40-q)^2$$

The airline then solves the following maximization problem, where t is the fraction of business travelers:

$$\max_q 450 - \frac{1}{2}(30-q)^2 + t\frac{1}{2}(40-q)^2 \tag{4}$$

which has solution $q^* = \frac{30-40t}{1-t}$. Prices are then $P_{coach} = A(q^*)$ and $P_{business} = A(q^*) + C(q^*)$. Clearly, the quality in coach is decreasing in t.

Problem 4 (30 points) Consider a market in which there are a continuum of used cars for sale. Each car is worth x to its current owner, where x is uniformly distributed on the unit interval across all cars. A car worth x to its seller is worth kx to a buyer, where $k > 1$. A buyer cannot observe x until after he purchases a car.

a. Suppose that sellers first decide whether or not to offer their cars for sale, and then those cars for sale are sold at a price halfway between the expected value to a seller of a car offered for sale and the expected value to a buyer of a car offered for sale. Find the equilibrium outcome of this market, as a function of k. Be sure to say what the price is and which values of x correspond to cars sold in equilibrium.

Suppose p is the price at which cars are sold. Clearly, only sellers who value their cars at less than p will participate in the market. Therefore, given that $x \sim U[0,1]$, the expected value of a car for sale is

$$E(x|x \leq p) = \min\{\frac{p}{2}, \frac{1}{2}\} \tag{5}$$

In order for p to be a price at which trade happens, we need

$$\frac{k+1}{2}\min\{\frac{p}{2}, \frac{1}{2}\} \leq \min\{p, 1\} \tag{6}$$

or $k \geq 3$. If $k < 3$, $p = 0$ no trade takes place. If $k > 3$, $p = \frac{k+1}{4}$, and all cars are sold. There are multiple equilibria in the knife-edge case of $k = 3$, but this is not important to the answer.

Now suppose $k = 4$. Suppose that each seller has three options: (i) not offering his car for sale, (ii) having the car subjected to a test at cost $c = \frac{5}{8}$ that perfectly and publicly reveals x and then offering the car for sale, or (iii) offering the car for sale without the test. Buyers are then randomly matched to sellers, and cars are sold at a price halfway between the actual value of the car to the seller and the actual value to the buyer (if the car has been tested), or halfway between the expected value of a randomly selected car to a seller and the expected value of an untested car to a buyer, if it has not been tested.

b. What is the net payoff to a seller of having his car tested, as a function of x?

A seller who has his car tested will be able to sell it for x, and so his payoff is $\frac{4+1}{2}x - x - c = \frac{3}{2}x - \frac{5}{8}$.

c. What is the net payoff to a seller of selling his car without testing, as a function of μ_x, the average value of x across all untested cars?

The expected payoff to selling a car without testing is $\frac{4+1}{2}\mu_x - x$.

d. Using parts b. and c., show that in any equilibrium, at least *some* sellers have their cars tested. Do this as follows: suppose all sellers sell their cars without testing, so that $\mu_x = \frac{1}{2}$. Show that, in this case, a seller with $x > \frac{3}{4}$ strictly prefers to have his car tested.

Suppose that all sellers sell their cars without testing, so that $\mu_x = \frac{1}{2}$. In this case, a seller of a car valued at x prefers to sell his car with the test iff

$$\frac{3}{2}x - \frac{5}{8} \geq \frac{5}{2} * \frac{1}{2} - x \tag{7}$$

which reduces to $x \geq \frac{3}{4}$. The claim follows.

e. Show that in equilibrium sellers with $x \geq \frac{1}{2}$ prefer to have their cars tested, while those with $< \frac{1}{2}$ prefer to sell their cars without the test.

Let $\underline{x}$ be the minimum value of x at which a seller sells his car with the test (we know from c. that $\underline{x} < 1$). Then, $\mu_x = \frac{\underline{x}}{2}$. Furthermore, a seller with a car valued at $\underline{x}$ must be indifferent between selling his car with the test and selling it without, or

$$\frac{3}{2}\underline{x} - \frac{5}{8} = \frac{5}{2} * \frac{\underline{x}}{2} - \underline{x} \tag{8}$$

which reduces to $\underline{x} = \frac{1}{2}$.

f. Going back to part a, who gains and who loses from allowing quality testing in this market? Are buyers *ex ante* better off or worse off? What about sellers? Does the answer for sellers depend on x?

Without testing (part a.), and given $k = 4$ all cars are sold at a price of $\frac{5}{4}$, so buyers get an *ex ante* payoff of $E(4x - \frac{5}{4}) = \frac{3}{4}$. With testing, buys who buy an untested car pay a price of $\frac{5}{8}$ and get expected surplus of $E[4x - \frac{5}{8} | x \in [0, \frac{1}{2}]] = \frac{3}{8}$. Those customers who buy a tested car pay a price of $\frac{5}{2}x$ and get an average surplus of $E[4x - \frac{5}{2}x] = \frac{9}{8}$. Given that a buyer is equally likely to be matched with a seller who tested his car and one who did not, a buyer's *ex ante* payoff is $\frac{3}{4}$. Intuitively, the same amount of trade takes place with or without testing, and a buyer gets on average half the surplus from a sale either way. Since a buyer's surplus is the same with or without testing, a seller's *ex ante* surplus must be lower, as his expected surplus from a sale is the same, but now he must pay for the cost of testing. Of course, a seller with $x = 1$ (for example) prefers to have the option of testing.