Homework 3

answers

Problem 1 MWG problem 9.C.2

First, if $\gamma < 0$, Out is not a dominated strategy for Firm E, as it was in the example we did in class.

Firm I plays fight only if $\mu_x \geq \frac{2}{3}$, where x is the left node in Firm I's information set. Consider the following three supports for Firm I:

- F: requires $\mu_x \geq \frac{2}{3}$. If Firm I plays F, Firm E plays Out. Any beliefs for Firm I are then consistent with Bayes' rule.
- A: requires $\mu_x \leq \frac{2}{3}$. If Firm I plays A, Firm E plays In_1, which means Firm I's belief is not consistent with Bayes' rule.
- mix: requires $\mu_x = \frac{2}{3}$. For this belief to be consistent, either Firm E plays Out or plays In_1 twice as often as In_2. Firm E is willing to play out so long as Firm I plays fight with probability greater than $\max\{\frac{3}{4}, \frac{2}{2-\gamma}\}$. Firm E is willing to mix so long as Firm I plays fight with probability $\frac{1}{2+\gamma}$.

PBE: (out, fight, $\mu_x \geq \frac{2}{3}$), (out, $\sigma_2(fight) \geq \max\{\frac{3}{4}, \frac{2}{2-\gamma}\}$, $\mu_x = \frac{2}{3}$), ($\frac{2}{3}In_1 + \frac{1}{3}In_2$, $\frac{1}{2+\gamma}$ Fight + $\frac{1+\gamma}{2+\gamma}$ Accomodate , $\mu_x = \frac{2}{3}$[1]

Problem 2 MWG problem 9.C.7

a: 1 plays B, 2 plays D at her left node, U at her right node. It is unique, though 1 playing T and 2 playing U at both nodes is also a Nash equilibrium.

b: T is now a dominant strategy for 1. The unique Nash equilibrium is T, U.

c: See final page for the picture of the extensive form game. Note that player 2 plays D at his first information set if $\mu_\alpha \geq \frac{1}{2}$ and at his second information set if $\mu_x \geq \frac{1}{2}$. Let q be the probability 1 plays B. Player 2 has 9 possible supports. Consider them one by one:

- D,D: 1 plays T and 2's beliefs are not consistent with Bayes' rule.
- U,U: 1 plays T, and 2's beliefs $\mu_\alpha = \mu_x = 0$ are consistent with Bayes rule. PBE.
- D,U: 2's optimality requires $q(1-p) \geq (1-q)p$ and $(1-q)(1-p) \geq qp$. These two equations being satisfied simultaneously requires $\frac{q}{1-q} \in [\frac{p}{1-p}, \frac{1-p}{p}]$, which means 1 must mix. 1 is indifferent only if $p = \frac{1}{3}$. If $p \neq \frac{1}{3}$, there is no equilibrium. If $p = \frac{1}{3}$, $q \in [\frac{1}{3}, \frac{2}{3}]$, and beliefs consistent with Bayes' rule comprise a PBE.
- U,D: Similarly to previous case, there is only an equilibrium if $p = \frac{2}{3}$ (with $q \in [\frac{1}{3}, \frac{2}{3}]$).
- mix, D: player 2's optimality requires $q(1-p) = (1-q)p$ and $qp \geq (1-q)(1-p)$. The former requires $q = p$, the latter $p \geq \frac{1}{2}$. For player 1 to be indifferent, 2 must play $\frac{6p-4}{6p-3}D + \frac{1}{6p-3}U$ at his left info (which is only sensible if $p \geq \frac{2}{3}$. So long as $p \geq \frac{2}{3}$, there is a PBE with 1 playing $q = p$, and 2 playing as above, with appropriate beliefs.

[1] In the event that $\gamma = -\frac{2}{3}$, Firm E is willing to mix over all three of his strategies.

- mix, U: These strategies are optimal for 2 only if $q = p$ and $p \leq \frac{1}{2}$. 1 is indifferent between B and T iff 1 plays $\frac{1}{3-6p}D + \frac{2-6p}{3-6p}U$. So long as $p \leq \frac{1}{2}$, there is a PBE with 1 playing $q = p$, and 3 playing as above, with appropriate beliefs.

- U, mix: 2 mixes at his right information set only if $q = p = \frac{1}{2}$, but in this case 1 strictly prefers T, in which case mixing is not optimal for player 2

- D, mix: 2 mixes at his right information set only if $q = p = \frac{1}{2}$, but in this case 1 strictly prefers T, in which case mixing is not optimal for player 2

- mix, mix: 2 mixes at his right information set only if $q = p = \frac{1}{2}$, but in this case 1 strictly prefers T, in which case mixing is not optimal for player 2

To summarize, the following are PBE:

- 1 plays T. 2 plays (U,U), and has beliefs $\mu_\alpha = \mu_x = 0$.

- If $p = \frac{1}{3}$, there is an equilibrium in which 1 plays $q \in [\frac{1}{3}, \frac{2}{3}]$, and 2 plays (D, U) and has beliefs $\mu_\alpha = \frac{2q}{1+q}$ and $\mu_x = \frac{q}{2-q}$.

- If $p = \frac{2}{3}$, there is an equilibrium in which 1 plays $q \in [\frac{1}{3}, \frac{2}{3}]$, and 2 plays (U, D) and has beliefs $\mu_\alpha = \frac{q}{2-q}$ and $\mu_x = \frac{2q}{1+q}$.

- If $p \geq \frac{2}{3}$, there is an equilibrium in which 1 plays $q = p$, and 2 plays $\frac{6p-4}{6p-3}D + \frac{1}{6p-3}U$ and has beliefs $\mu_\alpha = \frac{1}{2}$ and $\mu_x = \frac{p^2}{p^2+(1-p)^2}$.

- If $p \leq \frac{1}{2}$, there is an equilibrium in which 1 plays $q = p$, and 2 plays $\frac{1}{3-6p}D + \frac{2-6p}{3-6p}U$ with beliefs $\mu_\alpha = \frac{1}{2}$ and $\mu_x = \frac{p^2}{p^2+(1-p)^2}$.

The statement in the text of the question that there is a unique PBE seems to be unfounded, without additional parameter restrictions (i.e. $p \in (\frac{1}{2}, \frac{2}{3})$ or qualifying statements (i.e. unique *pure-strategy* equilibrium).

Problem 3 Ace-King-Queen poker is a two-card game that is played using a deck consisting of three cards: an ace (the high card), a king (the middle card), and a queen (the low card). Play proceeds as follows:

- Each player puts \$1 in a pot in the center of the table.

- The deck is shuffled, and each player is dealt one card. Each player sees only the card he is dealt.

- Player 1 chooses to raise (R) or fold (F). A choice of R means that player 1 puts an additional \$1 in the pot. Choosing F means that player 1 ends the game, allowing player 2 to have the money already in the pot.

- If player 1 raises, then player 2 chooses to call (c) or fold (f). A choice of c means that player 2 also puts an additional \$1 in the pot; in this case, the players reveal their cards and the player with the higher card wins the money in the pot.

a. Draw the extensive form of this game. See the figure on the final page of this file. Since it is such a large game, I have not attempted to use different letters for different behavior strategies, and I have omitted the probabilities nature assigns to each type profile ($\frac{1}{6}$, in every case).

b. Find all (weak) perfect Bayesian equilibria of this game.

Start with the easy parts: 2 folds if she has a queen, and calls when she has an ace. 1 raises when he has an ace. These are all dominant strategies

Given the above, 1 will raise when he has a king. As he regards it as equally likely that 2 has a queen or an ace should he have a king, his payoff to raising is $-2 * .5 + 1 * .5 = -.5$. As his payoff to folding is -1, he prefers raising to folding.

All that remains is to determine what 1 should do if he has a queen, and what 2 should do if she has a king. As usual, the way to do this is to consider all possible supports. Let's try looking at each possible move for player 2:

- 2 calls with a king. If this is the case, then 1 will always fold a queen. But in this case, 2 should believe that 1 has an ace when she has a king and her information set is reached, in which case folding is optimal. Therefore, there is no equilibrium in which 2 always calls when she has a king.
- 2 folds with a king. If this is the case, 1 will always raise with a queen, in which case 2 should think there is a 50/50 chance 1 has either an ace or a queen when she has a king, but in this case, 2 prefers to call with a king. Therefore, there is no equilibrium in which 2 always folds a king.
- 2 mixes when she has a king. For 2 to be willing to mix, her expected payoff to folding and calling must be equal. This, in turn, requires $\mu_2(A|k) = \frac{3}{4}$. For this to be a Bayesian belief, it must be that $\sigma_1(R|Q) = \frac{1}{3}$. For this to be optimal for player 1, he must be indifferent between R and F when he has a queen. This is true if $\sigma_2(c|k) = \frac{1}{3}$.

Since we did an exhaustive search for equilibria, we can conclude there is only one equilibrium, in which 1 raises with an ace, raises a king, and raises a queen $\frac{1}{3}$ of the time, and in which 2 calls with an ace, calls with a king $\frac{1}{3}$ of the time, and folds a queen. 1's beliefs are 50/50 at each of his info sets. 2 has beliefs $\mu_2(K|a) = \frac{3}{4}$, $\mu_2(A|k) = \frac{3}{4}$, and $\mu_2(A|q) = .5$.

c. If you could choose to be either player 1 or player 2 in this game, which player would you choose to be?

Each type profile (Ak, Aq, Ka, Kq, Qa, Qk) is equally likely. 1's expected payoff is then

$$\frac{1}{6}\left((\frac{1}{3} * 2 + \frac{2}{3} * 1) + 1 + (-2) + 1 + (\frac{1}{3} * (-2) + \frac{2}{3} * (-1)) + (\frac{2}{3} * (-1) + \frac{2}{9} * 1 + \frac{1}{9} * (-2) \right) = -\frac{1}{9}$$

2's expected payoff is $\frac{1}{9}$ (you can compute this directly, or note that this is a zero-sum game, and so necessarily 2's payoff is the negative of 1's). Therefore, you are better off being the second mover in this game.

Problem 4 Solve for all (weak) perfect Bayesian equilibria in the game depicted in figure 1.

Obvious parts: S^2 is dominant for 2, and $\mu_w = \frac{19}{20}$. 1 prefers C^2 only if $\mu_y \leq \frac{4}{5}$.

Suppose first that 1 plays C^1 with positive probability. Then 1's beliefs at his second information set are calculated via Bayes rule: $\mu_y = \frac{19\sigma_2(c^1)}{19\sigma_2(c^1)+1}$. Consider each of three supports for player 1 at his second information set:

- C^2: requires $\mu_y \leq \frac{4}{5}$, which, via Bayes rule, requires $\sigma_2(c^1) \leq \frac{4}{19}$. But if 1 plays C^2, 2 strictly prefers c^1 to s^1, and so will always play c^1. No equilibrium.
- S^2: requires $\mu_y \geq \frac{4}{5}$, which, via Bayes rule, requires $\sigma_2(c^1) \geq \frac{4}{19}$. But if 1 plays S^2, 2 strictly prefers s^1 to c^1, and so will always play s^1. No equilibrium.

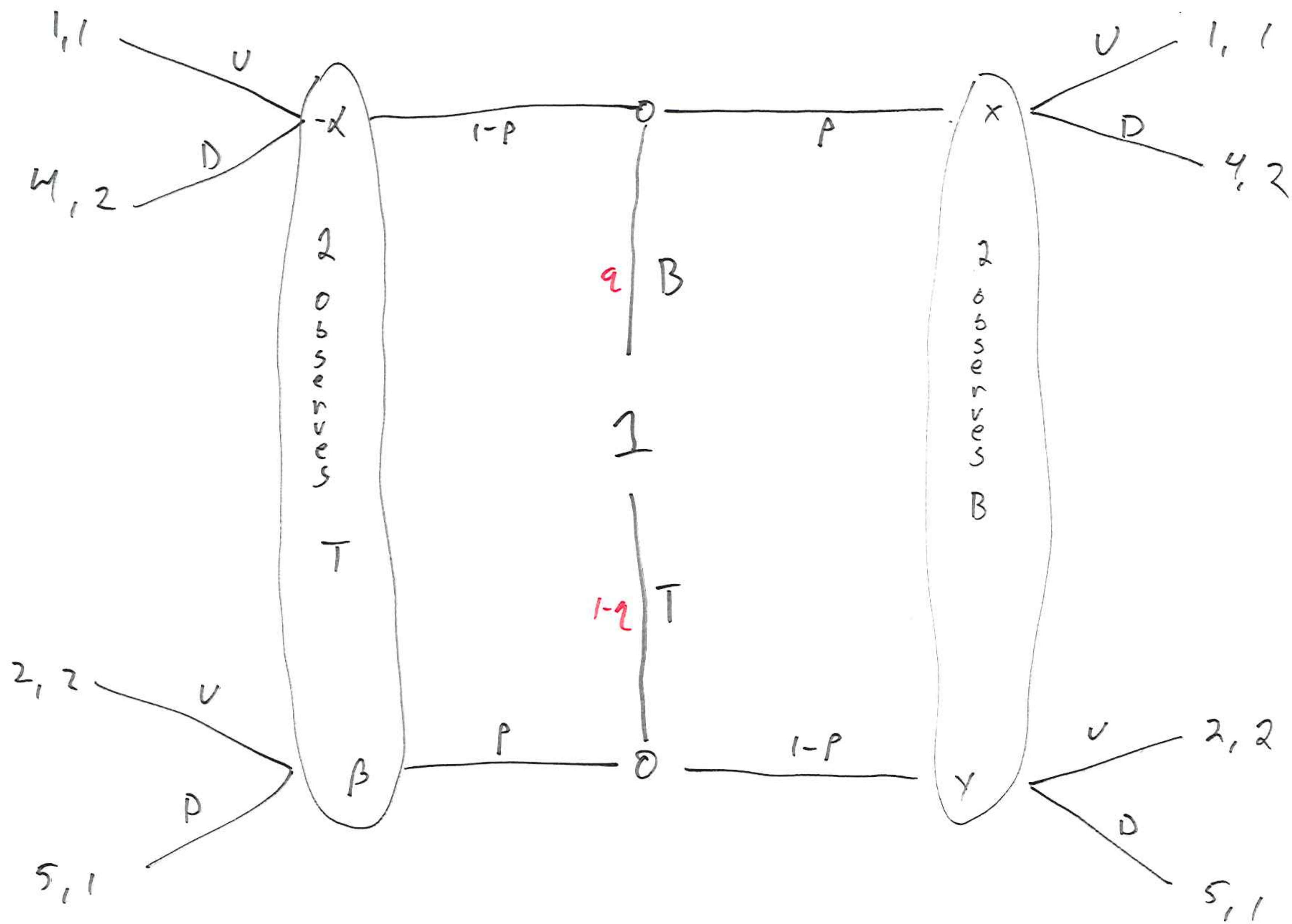
- mix: requires $\mu_y = \frac{4}{5}$, and so $\sigma_2(c^1) = \frac{4}{19}$. For 2 to be indifferent, 1 must play $\frac{1}{5}C^2 + \frac{4}{5}S^2$. In this case, 1 also strictly prefers C^1 to S^1. Equilibrium.

Finally, suppose 1 plays S^1 with probability 1. For this to be optimal, 2 must (at least weakly) prefer s^1 to c^1, which means 1 must put weight of at least $\frac{4}{5}$ on S^2.

In summary, the PBE of this game are as follows:

- 1 plays C^1 and $\frac{1}{5}C^2 + \frac{4}{5}S^2$, 2 plays $\frac{4}{19}c^1 + \frac{15}{19}s^1$ and s^2. $\mu_w = \frac{19}{20}$ and $\mu_y = \frac{4}{5}$.
- 1 plays S^1 and plays S^2 with probability greater than or equal to $\frac{4}{5}$, 2 plays s^1 and s^2, and $\mu_w = \frac{19}{20}$ and $\mu_y \geq \frac{4}{5}$ ($\mu_y = \frac{4}{5}$ if $\sigma_1(S^2) < 1$).
- 1 plays S^1 and plays S^2 with $\frac{4}{5}$ probability, 2 plays $\sigma_2(s^1) \geq \frac{19}{24}$ and $\sigma_2(s^2) = 1$, and $\mu_w = \frac{19}{20}$ and $\mu_y = \frac{4}{5}$

Note that the last two are not sequential equilibria (it is easy to show this via the parsimony criterion.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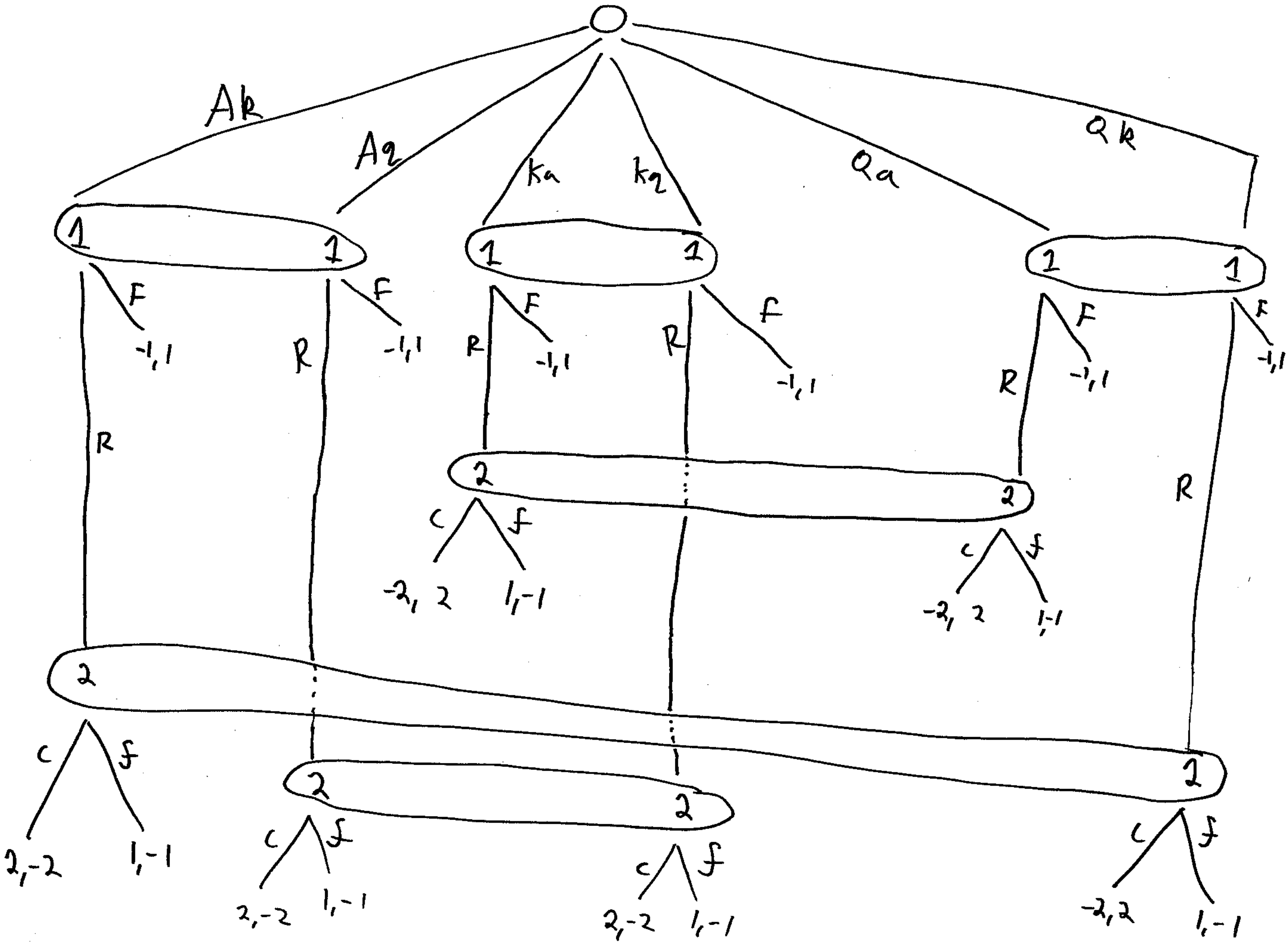
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