Homework 5

answers

Problem 1 Suppose market demand is given by $p(q) = a - bq$, and there are two firms, each with a constant marginal costs of c and no fixed cost. The two firms choose quantity simultaneously, and then sell whatever they have produced at the prevailing market price.

a. Determine NE quantities for both firms. Demonstrate that there is only one equilibrium in this game.

Firm i's best response function is $q_i = \frac{a-c}{2b} - \frac{q_{-i}}{2}$, for $i = 1, 2$. Clearly, this is linearly decreasing in q_{-i}, and so there can be at most one Nash equilibrium (draw a picture to convince yourself of this). The unique Nash equilibrium is located at $(q_1, BR_2(q_1))$ satisfying $q_1 = BR_1(BR_2(q_1))$, or $q_1 = q_2 = \frac{a-c}{3b}$.

b. Derive the market price, and the profit for each firm. Show that the total quantity produced is greater than the monopoly quantity, but less than the competitive quantity.

The market price is $\frac{a+2c}{3}$, while each firm earns a profit of $\pi^c = \frac{(a-c)^2}{9b}$.

c. Now suppose that firm 1 moves first. Firm 2 moves only after observing the quantity firm 1 chooses. Derive the SPE of this game.

Now firm 1's maximization problem changes to the following:

$$\begin{aligned} &\max_{q_1}(a - bq_1 - b(\frac{a-c}{2b} - \frac{q_1}{2}))q_1 - cq_1 \\ =&\max_{q_1}(\frac{a-c}{2} - b\frac{q_1}{2})q_1 \end{aligned}$$

which has FOC $q_1^s = \frac{a-c}{2b}$. Plugging this into 2's best response function gives $q_2^s = \frac{a-c}{4b}$. The market price is $p^s = \frac{1}{4}a + \frac{3}{4}c$, firm 1's profit is $\pi_1^s = \frac{(a-c)^2}{8b}$, and firm 2's profit is $\pi_2^s = \frac{(a-c)^2}{16b}$.

d. Finally, suppose there are J firms serving the market (and choosing quantity simultaneously). In the static case, determine NE quantities and profits for each of the J firms. Show that as $J \to \infty$, total production approaches the competitive level, while when $J = 1$, we get the monopoly outcome.

The Nash equilibrium is for each firm to produce $q = \frac{a-c}{b(J+1)}$, leading to a market price of $a - (a-c)\frac{J}{J+1}$ and per firm profit of $\frac{(a-c)^2)}{(J+1)^2 b}$. Clearly, as $J \to \infty$, price converges to c, per-firm profits converge to zero, and total quantity supplied converges to $\frac{a-c}{b}$, the efficient quantity.

Problem 2 An inventor has discovered a new method of producing a precious stone, using spring water found only in Venice, Italy and Danville, Kentucky. The process is patented and manufacturing plants are set up in both places. The product is sold only in Europe and the US. Trade laws are such that the price must be uniform within Europe and the US, but the European and American prices may differ. Transport costs are negligible, and there is no second-hand market in the stones because of the risk of forgeries. From the production and marketing data given below, determine the profit-maximizing production and sales plans. In particular, determine the output in Venice and Danville, sales in the US and Europe, quantity shipped from Europe to the US or vice versa, and prices in Europe and the US.

Demand: US, $p = 1500 - \frac{1}{2}Q$; Europe, $p = 1000 - Q$

Average cost: Danville, $AC = 150 + .375Q$; Venice, $AC = 100 + \frac{1}{2}Q$

A profit-maximizing monopolist will produce so that

$$MR_{US} = MR_E = MC_{US} = MC_E \tag{1}$$

Market clearing demands that

$$q^s_{US} + q^s_E = q^d_{US} + q^d_E \tag{2}$$

Solving this system yields $q^s_{US} = 600$ $q^s_E = 500$, $q^d_{US} = 900$, and $q^d_E = 200$. The price in the US will be $\$1,050$, and the price in Europe $\$800$. Europe will export 300 stones to the US.

Problem 3 MWG, problem 12.D.3

a. On the equilibrium path, each duopolist earns a profit of $\frac{(a-c)^2}{8b}$. If a duopolist plays his stage game best response, he will play $\frac{3}{8}\frac{a-c}{b}$ and earn a profit of $\frac{9}{64}\frac{(a-c)^2}{b}$. On the punishment path (Nash reversion), each duopolist earns the Cournot payoff of $\frac{(a-c)^2}{9b}$. The duopolist's incentive constraint for not deviating is then given by:

$$\frac{(a-c)^2}{8b} \geq (1-\delta)\frac{9}{64}\frac{(a-c)^2}{b} + \delta\frac{(a-c)^2}{9b} \tag{3}$$

which reduces to $\delta \geq \frac{9}{17}$.

b. Each duopolist earns an equilibrium path profit of $(a - 2bq - c)q$. The maximal stage game payoff to deviating from the equilibrium path is $(\frac{a-c}{2} - \frac{1}{2}bq)(\frac{a-c}{2b} - \frac{1}{2}q)$. The (Nash reversion) punishment path gives each duopolist the Cournot payoff of $\frac{(a-c)^2}{9b}$. The duopolist's incentive constraint for not deviating is then given by:

$$\begin{aligned}(a-2bq-c)q &\geq (1-\delta)(\frac{a-c}{2} - \frac{1}{2}bq)(\frac{a-c}{2b} - \frac{1}{2}q) + \delta\frac{(a-c)^2}{9b} \\ \Rightarrow \delta &\geq 1 - \frac{(a-2bq-c)q}{(\frac{a-c}{2} - \frac{1}{2}bq)(\frac{a-c}{2b} - \frac{1}{2}q) - \frac{(a-c)^2}{9b}}\end{aligned} \tag{4}$$

it is straightforward to verify that the RHS of (4) is decreasing in q (MWG appears to be wrong in what it is asking you to show. Also, it's not really straightforward to show this, but you get the idea).

Problem 4 MWG, problem 12.E.4

Since the firms form a cartel no matter how many firms enter the market, industry output and price will be unchanged by the number of firms in the industry. Only the aggregate fixed costs increase as the number of firms increase. Therefore, the optimal number of firms in this industry is one.

If the planner cannot control entry, the equilibrium number of firms will be $J^* = \frac{\pi^m}{K}$. In terms of welfare this means that free entry leads to a complete dissipation of monopoly profits, without any benefit to consumers.

Problem 5 Consider an oligopoly with K firms. Each firm can produce costlessly, and market demand is given by $P = 1 - \sum_{i=1}^{K} q_i$.

a. Suppose $K = 3$, and the three firms are Cournot competitors. Solve for the Nash equilibrium profits of firms 1, 2, and 3 in this game.

Each firm will produce a quantity of $q = \frac{1}{4}$, the market price will be $P = \frac{1}{4}$, and each firm will earn profit of $\frac{1}{16}$.

b. Suppose $K = 3$. Suppose that firm 1 publicly commits to a quantity. After observing firm 1's choice of quantity, firms 2 and 3 simultaneously choose quantities. Solve for the subgame perfect equilibrium profits of firms 1, 2, and 3 in this game.

Firms 2 and 3 will each produce quantity $\frac{1-q_1}{3}$. Given this, firm 1 chooses q_1 to solve the following:

$$\max_{q_1}[1 - q_1 - \frac{2}{3}(1 - q_1)]q_1 \tag{5}$$

which has solution $q_1 = \frac{1}{2}$, meaning that firms 2 and 3 each produce quantity $\frac{1}{6}$, the market price is $\frac{1}{6}$, firm 1's profit is $\frac{1}{12}$, and firms 2 and 3 each earn profit of $\frac{1}{36}$.

For parts c-e: Suppose that K firms choose quantity sequentially (that is, firm 1 publicly announces a quantity. After observing q_1, firm 2 publicly announces a quantity. After observing q_1 and q_2, firm 3 publicly announces a quantity, and so on).

c. Suppose $K = 3$. Solve for each firm's profit in the subgame perfect equilibrium of this game.

Firm 3 will produce $\frac{1}{2} - \frac{1}{2}(q_1 + q_2)$. Given this, 2 chooses q_2 to solve:

$$\max_{q_2}(1 - q_1 - q_2 - (\frac{1}{2} - \frac{1}{2}(q_1 + q_2))q_2$$

which has solution $q_2 = \frac{1}{2} - \frac{1}{2}q_1$. Finally, given this, 1 chooses q_1 to solve:

$$\max_{q_1}[1 - q_1 - (\frac{1}{2} - \frac{1}{2}q_1) - (\frac{1}{4} - \frac{1}{4}q_1)]q_1 \tag{6}$$

which has solution $q_1 = \frac{1}{2}$. Therefore, $q_2 = \frac{1}{4}$ and $q_3 = \frac{1}{8}$. Market price is $\frac{1}{8}$, and profits are $(\pi_1, \pi_2, \pi_3) = (\frac{1}{16}, \frac{1}{32}, \frac{1}{64})$.

d. Now suppose K is a large number. Solve for the SPE of the game using the method of your choosing. Calculate firm k's SPE profits.

beginequation **Guess:** q$_k = \frac{1-\sum_{i=1}^{k-1} q_i}{2}$(6)

Verify this guess (think about what this guess is saying. Each firm produces enough quantity to satisfy half the remaining demand on its turn.)

The guess defines q_k recursively:

$$\begin{aligned}
q_2 &= \frac{1}{2} - \frac{1}{2}q_1 \\
q_3 &= \frac{1}{2} - \frac{q_1}{2} - \frac{\frac{1}{2} - \frac{q_1}{2}}{2} = \frac{1}{4} - \frac{1}{4}q_1 \\
q_4 &= \frac{1}{2} - \frac{q_1}{2} - \frac{\frac{1}{4} - \frac{1}{4}q_1}{2} = \frac{1}{8} - \frac{1}{8}q_1 \\
q_5 &= \frac{1}{2} - \frac{q_1}{2} - \frac{\frac{1}{4} - \frac{1}{4}q_1}{2} - \frac{\frac{1}{8} - \frac{1}{8}q_1}{2} = \frac{1}{16} - \frac{1}{16}q_1 \\
&\dots \\
q_k &= \frac{1}{2^{k-1}} - \frac{1}{2^{k-1}}q_1
\end{aligned}$$

Looking for a pattern like this is fine; you can also do this more formally by careful substitutions. For step

2- Firm 1 then chooses q_1 to solve:

$$\max_{q_1}(1 - q_1 - \sum_{k=2}^{K} q_k)q_1$$

$$\text{simplifying, } \max_{q_1}(1 - q_1 - \sum_{k=1}^{K-1} \left(\frac{1}{2}\right)^k (1 - q_1))q_1$$

$$\text{simplifying further, } \max_{q_1}(1 - q_1 - (1 - \frac{1}{2}^{K-1})(1 - q_1))q_1$$

$$\text{simplifying still further, } \max_{q_1} \frac{1}{2}^{K-1}(1 - q_1)q_1$$

which has solution $q_1 = \frac{1}{2}$, as was to be shown. (note: it does not immediately follow from this that firm 2 is optimizing by playing $\frac{1}{2} - \frac{1}{2}q_1$. The best way to show this would be to use a similar proof to show that if demand is $a - \sum_{k=1}^{K} q_k$ then firm 1 optimally plays $\frac{a}{2}$, and then it does follow that $q_2 = \frac{1}{2} - \frac{1}{2}q_1$.

Profit is given by

$$\pi_k = \frac{1}{2}^K * \frac{1}{2}^k = \frac{1}{2}^{K+k}$$