Homework 7

answers

Problem 1 (January 2012 prelim) A firm has two types of jobs, good jobs and bad jobs. When a *qualified* worker is assigned to a good job, the firm earns a net profit of $20,000. When an *unqualified* worker is assigned to a good job, the firm incurs a net loss of $20,000. When a worker of either type is assigned to a bad job, the firm breaks even. Workers prefer good jobs, and get an extra $32,000 payoff from a good job relative to a bad job.

To become qualified, a worker pays an investment cost c. This cost is higher for some workers than for others; the distribution of c across all workers is uniform between $0 and $9,000. The firm cannot observe which workers are qualified and which are not.

a. If the firm has no additional information about new workers, how many workers become qualified in the equilibrium of the model?

If the firm cannot tell which workers are qualified and which are not, then their decision to place a worker into a good job is independent of whether a worker actually is qualified or not. In this case, no worker has an incentive to become qualified, so all workers willbe unqualified.

Now suppose that while the firm cannot directly observe workers' investment decisions, it administers a test to new employees, with scores ranging from 0 to 1. The probability a qualified worker scores less than $t \in [0,1]$ is t^2. The probability an unqualified worker scores less than t is t.

b. Suppose that the firm puts all workers with a test score of $s \in [0,1]$ or higher into a good job. Describe the incentive constraint for a worker's decision to become qualified or not. What fraction π of workers will become qualified, as a function of s?

The benefit to becoming qualified is $32,000(s - s^2)$. The cost is c. Given $c \sim U[0, 9000]$, the fraction of workers who become qualified, as a function of s, is

$$\pi = \frac{32}{9}s(1-s) \tag{1}$$

c. Now consider the firm's problem. Suppose that fraction π of all workers become qualified. Show that the firm optimally puts workers scoring above some cutoff test score s into good jobs, and puts low-scoring workers into bad jobs, and solve for s as a function of π.

The firm's posterior belief that a given worker who received test score θ is qualified is $p(\theta) = \frac{\pi 2\theta}{\pi 2\theta + 1 - \pi}$. The firm will put the worker into the good job iff $p * 20,000 - (1-p)20,000 \geq 0$. Simplifying, the firm will put a worker into a good job iff $\pi \geq \frac{1}{1+2\theta}$. The firm's cutoff test score is determined by where this holds with equality, or

$$s = \frac{1-\pi}{2\pi} \tag{2}$$

d. An equilibrium is (s, π) pair such that s maximizes firm profit given π and π is consistent with workers maximizing expected wages net of the investment cost given s. Characterize the equilibrium values of π and s as follows. One, show that $s = \frac{1}{4}$ and $\pi = \frac{2}{3}$ is an equilibrium. Two, show (don't try to solve for it explicitly!) that there is another equilibrium with $s > \frac{1}{4}$.

The pair $(\pi, s) = (\frac{2}{3}, \frac{1}{4})$ clearly satisfies both (5) and (2), as required for an equilibrium. A picture helps to show that there is a second equilibrium. See the figure at the end of this answer set. Equation (1) is a concave function maximizes at $s = \frac{1}{2}$ and equal to zero at $s = 0$ and $s = 1$. Equation (2) is a downward sloping function that crosses the red line at $s = \frac{1}{4}$ and is always positive. This implies that there must be a second equilibrium at $s_2^* > \frac{1}{4}$.

e. What economic interpretation does the Coate and Loury paper studied in class assign to the multiplicity of equilibria in its model?

The possibility of rational discrimination; "bad" equilibria correspond to discriminated-against groups, and "good" equilibria correspond to favored groups.

Problem 2 (spring 2011 final) Consider a market with two or more firms and a continuum of workers. Each firm has two types of jobs, "old" jobs and "new" jobs. The profit to the firm and the payoff to the worker, when the worker is assigned to an old job, is 0. The payoff to a worker assigned to a new job is 1. The payoff to a firm when assigning the worker to the new job is 1 if the worker is skilled, and -1 if the worker is not skilled (all payoffs already include wages). A worker must pay a cost of c to acquire skills. The value of c differs across different workers, with c being uniformly distributed on $[0, 1]$.

a. Suppose that workers first decide whether to acquire skills and then are matched to firms, who assign them to jobs. Suppose that the firms *can* observe whether each worker has acquired skills. Find the pure-strategy equilibrium job-assignment and skill-acquisition decisions.

If investment decisions are observed, firms will place anyone who has acquired skills into a new job, and anyone who has not into an old job. Given this, all workers will acquire skills.

b. Now suppose that workers first decide whether to acquire skills and then are matched to firms, who assign them to jobs. Suppose that firms *cannot* observe whether a worker has acquired skills. Find the pure-strategy equilibrium and skill-acquisition decisions.

If firms assign everyone to new jobs, no one will invest, as job placement is independent of skill acquisition. If firms assign all workers to old jobs, again, no one will invest. Therefore, the only equilibrium is for no worker to acquire skills, and for firms to place all workers into old jobs.

c. Now suppose that workers first decide whether to acquire skills, then take a test, and then are matched to firms, who assign them to jobs. Firms cannot observe whether a worker has acquired skills, but can observe the outcome of the test, which is either a pass (p) or fail (f).[1] A worker who has acquired skills passes the test with probability $\alpha > \frac{3}{4}$ and fails with probability $1 - \alpha$, while a worker who has not acquired skills passes with probability $1 - \alpha$ and fails with probability α. Find the equilibrium job-assignment and skill-acquisition decisions. (There are multiple such equilibria. Find all the pure strategy equilibria first. Consider mixed strategy equilibria if time permits.)

[1] This is a similar setup to a model studied in class, but note that here the test has only two possible outcomes, whereas in class, the test score was continuously measured.

There are three possible firm actions in a pure strategy equilibrium: they can place all workers in new jobs, place all workers in old jobs, or place only workers who pass the test into new jobs. If they place all workers into new jobs, then no worker will acquire skills, and so the firm would clearly be better off placing workers into old jobs, and this is not an equilibrium.

If firms place all workers into old jobs, no worker wants to acquire skills, and so firms are justified in putting all workers into old jobs (and firms believe that fraction $\pi = 0$ of workers are qualified, and continue to believe this after either test result).

If firms promote only workers who pass the test, then a worker will acquire skills if:

$$\alpha - c \geq 1 - \alpha$$
$$c \leq 2\alpha - 1 \tag{3}$$

and so fraction $2\alpha - 1$ of all workers acquire skills under this scenario. To check to see if this is an equilibrium, note that in any equilibrium a firm's belief prior to seeing the test result that a given worker is qualified is $2\alpha - 1$. If the worker passes the test, this updates to a posterior of:

$$\mu_p = \frac{(2\alpha - 1)\alpha}{(2\alpha - 1)\alpha + 2(1 - \alpha)^2}$$
$$= \frac{2\alpha^2 - \alpha}{4\alpha^2 - 5\alpha + 2} \tag{4}$$

In order for the firm's strategy to be optimal, it must be that $\mu_p \geq \frac{1}{2}$ and $\mu_f \leq \frac{1}{2}$. Checking the first condition from (4) gives that $\mu_p \geq \frac{1}{2}$ iff $\alpha \geq \frac{2}{3}$. An analogous condition gives that $\mu_f \leq \frac{1}{2}$ for all $\alpha \geq \frac{3}{4}$, and so the firms are optimizing given worker behavior. This is an equilibrium. I will not take the time to solve for mixed strategy equilibria here, but this would be a good exercise to prepare for prelims.

d. Now suppose that workers come in two varieties, red and green. The colors have no effect on the cost of acquiring skills, test outcomes, the value of acquiring skills, or anything else, but are observed by firms. Is there an equilibrium in which different colored workers behave differently?

Yes. There are two equilibria in part c, one in which firms have a prior of 0 and never promote anyone, and one in which firms have a prior of $2\alpha - 1$ and promote all workers who pass the test. Apply one equilibrium to red workers, the other to green workers, and we have a discriminatory equilibrium, even though red and green workers are *ex ante* identical.

Problem 3 Consider an economy in which there are equal numbers of men and women, and two kinds of jobs, good and bad. Some workers are qualified for the good job, and some are not. Employers believe that the proportion of men who are qualified is $\frac{2}{3}$ and the proportion of women who are qualified is $\frac{1}{3}$. If a qualified worker is assigned to the good job, the employer gains \$1,000, while if an unqualified worker is assigned to the good job, the employer loses \$1,000. When any worker is assigned to the bad job, the employer breaks even.

Workers who apply for jobs are tested and assigned to the good job if they do well on the test. Test scores range from 0 to 1. The probability that a qualified worker will have a test score less than t is t. The probability that an unqualified worker will have a test score less than t is $t(2 - t)$. Employers are subject to a rule that requires the proportion of men assigned to the good job to be the same as the proportion of women. Otherwise, employers maximize expected profits.

a. Find the profit-maximizing policy for an employer. Note that in this problem we take as given employer attitudes towards men and women; they do not need to be determined endogenously.

As the fraction of men and women who are qualified (π_M and π_W, respectively) is given exogenously, employers choose only s_A and s_B subject to two constraints. One, a firm must be indifferent between putting a randomly drawn worker who just barely passed the test into the good job or the bad job ((5) below). Two, s_A and s_B must satisfy the affirmative action constraint, which says that the overall proportion of men and women promoted to the good job must be the same ((6) below).

$$\pi_M(1-F_q(s_A))+(1-\pi_M)(1-F_u(s_A))=\pi_W(1-F_q(s_B))+(1-\pi_W)(1-F_u(s_B)) \quad (5)$$

$$\frac{x_u}{x_u+x_q}=\frac{\frac{1}{2}}{1+\frac{1-\pi_M}{\pi_M}\frac{f_o(s_A)}{f_q(s_A)}}+\frac{\frac{1}{2}}{1+\frac{1-\pi_W}{\pi_W}\frac{f_o(s_B)}{f_q(s_B)}} \quad (6)$$

Making the substitutions $\pi_M=\frac{2}{3}$, $\pi_W=\frac{1}{3}$, $F_q(s)=s$, and $F_u(s)=s(2-s)$, these reduce to:

$$2s_B^2-5s_B=s_A^2-4s_A \quad (7)$$

$$s_B=\frac{3-4s_A}{4-4s_A} \quad (8)$$

Solve this system using your prefered method. The profit-maximizing policy for the firm is $s_A=.54$ and $s_B=.456$.

b. Test your policy as follows. If you are told that a worker has just barely passed the test (and you are not told whether the worker is male or female), what is the probability that the worker is qualified? Is it the case that such a worker is a fair bet from the employer's point of view? If not, should the policy be adjusted?

Simply put the values of s_A and s_B you solved for back into the fair bet equation, and verify that each side gives you $\frac{1}{2}$. This should tell you that an employer who knows that a workers has just barely passed the test, but whose gender is unknown, is a fair bet, in that the employer is indifferent between assigning such a worker to a good job or a bad job.

Problem 4 Suppose that business travelers have marginal willingness to pay $40-q$ for a seat of quality $q\in[0,40]$, meaning that their total willingness to pay for a seat of quality $\hat{q}\in[0,40]$ is $\int_0^{\hat{q}}(40-q)dq$ (assume that marginal willingness to pay is 0 for $q>40$). Tourists have marginal willingness to pay of $30-q$ for $q\in[0,30]$, meaning their total willingness to pay for a seat of quality $\hat{q}\in[0,30]$ is $\int_0^{\hat{q}}(30-q)dq$ (assume tourists have marginal willingness to pay of 0 for $q>30$). Assume that 80 tourists and 20 business travelers typically fly a given route, and the the plane used on this route is more than big enough to hold all 100 travelers, so the airline never has to worry about a capacity constraint. However, the airline cannot tell which type a given traveler is, and so cannot condition price on group membership.

Suppose the airline is able to put two sections on the plane (i.e. 1st class and coach), each with its own quality level. Assume that the cost of setting quality level q in coach is K_c*q and that the cost of setting quality q in 1st class is $K_{fc}*q$, for $K_{fc}\geq K_c$.

a. For parts a-d, set $K_{fc}=K_c=0$. Suppose the airline sets $q=30$ in coach and $q=40$ in 1st class. Solve for the profit maximizing prices, taking these quality levels as given.

It helps to draw a picture with this problem. They will charge coach customers their full willingness to pay of \$450, and business travelers \$500, leaving them 300 surplus, the same amount they would get from buying a coach ticket (in order to incentivize business travelers to buy a 1st class ticket instead of coach. Note that it is also an option to set the price of business class to be \$800, the price of coach to be \$10M (so no one buys a coach ticket), and sell only to business travelers. Since the business travelers are a small percentage of all the travelers here, it is easy to see that this option is not optimal in this case.

b. You are hired as a consultant to advise the airline on how it can increase profits. Explain why decreasing the quality in coach — and in turn decreasing the price — can increase the airline's profit, even if the number of passengers flying the route remains 100, with 80 tourists and 20 business travelers.

Suppose the airline lowers the quality in coach to 28. Then, the price of a coach ticket is lowered to \$448, while the price of a business class ticket is raised to \$520. Thus, 20% of travelers are now paying \$20 more relative to part a, while the remaining 80% are paying only \$2 less, and so profits are higher. The idea is that the lower the quality in coach, the higher the price in 1st class, as the incentive constraint for business travelers has less bite.

c. Solve for the profit-maximizing price and quality levels in both coach and business class.

Maximizing profits means choosing four variables, p_c, p_{FC}, q_c, q_{FC} to maximize $80 * p_c + 20 * p_{BC}$ subject to an incentive constraint for business travelers and an individual rationality constraint for tourists.

First, clearly $q_{FC} = 40$. Second, individual rationality implies $p_c = 450 - \frac{1}{2}(30 - q_c)^2$. Third, the business travelers' incentive constraint implies $p_{FC} = p_c + \frac{1}{2}(40 - q_c)^2$. Therefore, the airlines profit maximizing coach quality level is given by the following unconstrained maximization problem:

$$\max_{q_c} 80 * (450 - \frac{1}{2}(30 - q_c)^2) + 20 * (p_c + \frac{1}{2}(40 - q_c)^2) \tag{9}$$

(drawing a picture will help quite a bit with determining what prices are implied by the constraints). Evidently, the maximizer of the above is $q_c^* = 27.5$, which implies the optimal coach price is \$446.87 and the optimal first class price is \$525.

d. Now suppose that the composition of travelers changes, so that fraction t of all travelers are business travelers, and fraction $1-t$ are tourists (the plane is still plenty big enough to hold all travelers, so constraints like there needing to be more seats in coach than there are passengers are not binding). Solve for the optimal price and quantity levels in coach and 1st class, as a function of t.

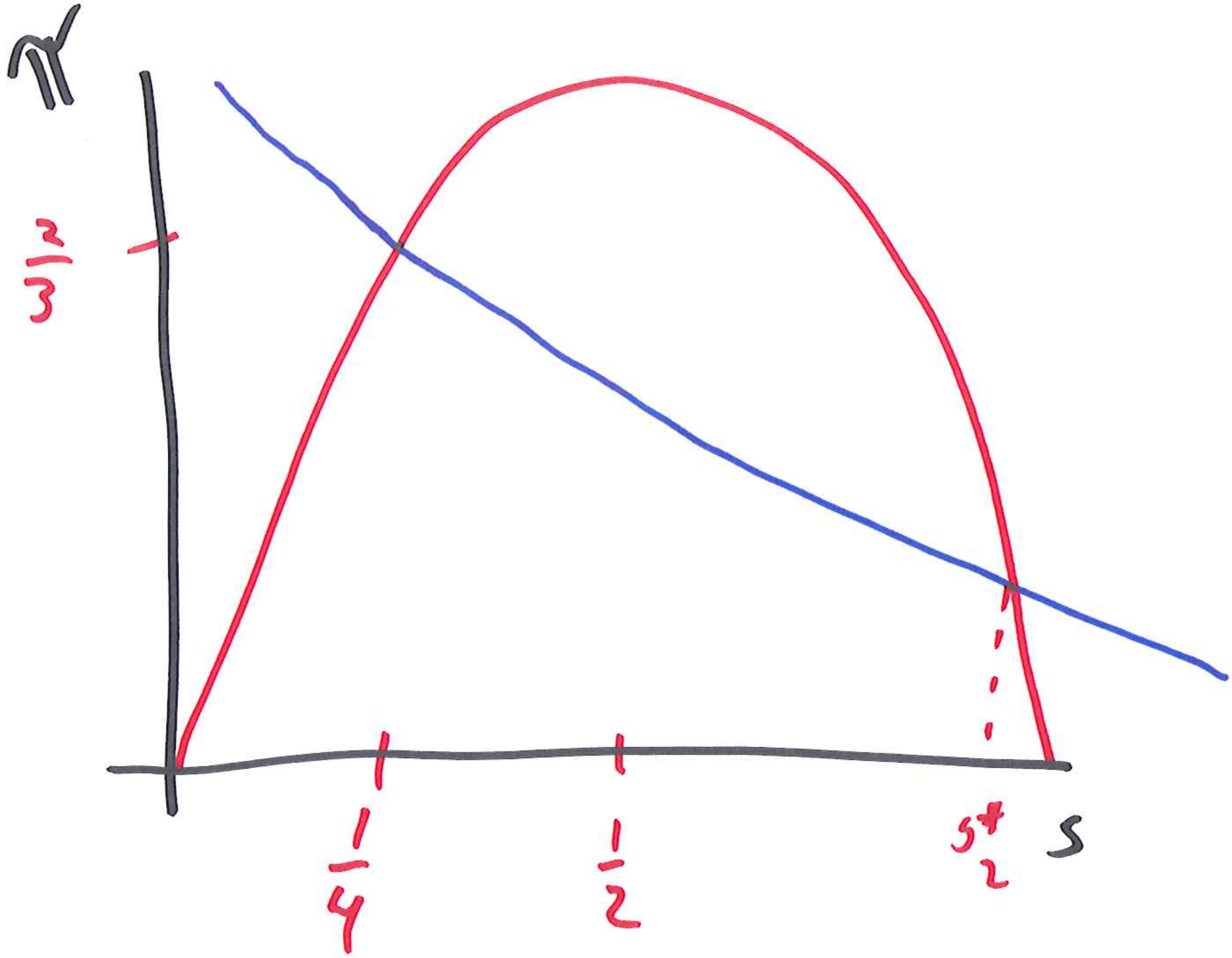
Same setup as in the previous subsection, except we maximize $(1-t)p_c + tp_{FC}$. The solution is:

$$q_c^* = \max\left\{\frac{30 - 40t}{1 - t}, 0\right\} \tag{10}$$

with prices set accordingly. Note that if the fraction of business travelers is over $\frac{3}{4}$, the airline optimally sells only to business travelers.

e. Finally, suppose that $K_c = 1$ and $K_{fc} = \$K$. Suppose again that there are 80 tourists and 20 business travelers. Solve for the relationship between the price of coach and K, and give an intuitive explanation for why these two variables are related in this way.

A high K would lower the quality set in first class, but this would not change the price or quality set in coach (note that K_c being one will lower the coach quality slightly relative to the case where $K_c = 0$). Again, the easiest way to see this is to draw a picture (see final page). As K_{fc} increases, q_{fc}^* will decrease from 40, but the airline's calculation in determining optimal q_c is unaffected, and so price and quality in coach are unchanged.



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