Midterm exam I

answers

Instructions: Throughout, points will be deducted mercilessly for insufficiently supported answers. When in doubt, err on the side of maximum verbosity. You may use books, notes, and calculators, but no other electronic devices. You may not discuss the exam with anyone other than me until all students have turned in their exams. Do not assume questions are ordered with respect to difficulty.

Problem 1 (25 points) Alistair and Braxton are playing the following simultaneous-move game:

		Braxton		
		x	y	z
	A	$2,0$	$1,1$	$1,0$
Alistair	B	$-1,2$	$4,0$	$0,0$
	C	$1,0$	$2,0$	$2,2$

a. Find all Nash equilibria of this game (pure and mixed). Support your answer.

See figure 1 for a picture of the best response correspondences. Consider all supports for player 1:

- A: 2 plays y, to which B is a best response. Not Nash.
- B: 2 plays x, to which A is a best reponse. Not Nash.
- C: 2 plays z, to which C is a best response. Nash equilibrium: (C,z).
- AB: 2 plays y, x, or yx. If 2 plays a pure strategy, 1 prefers a pure strategy. If 2 plays $\frac{1}{2}x + \frac{1}{2}y$, 1 is indifferent between A and B. Nash equilibrium: $(\frac{2}{3}A + \frac{1}{3}B, \frac{1}{2}x + \frac{1}{2}y)$.
- AC: 2 plays y, z, or yz. AC is not a best response to any of these for 1. Not Nash.
- BC: 2 plays x, z, or xz. BC is not a best response to any of these for 1. Not Nash.
- ABC: This is a best response for 1 if and only if 2 plays $\frac{1}{2}x + \frac{1}{2}y$. This is a best response for 2 if and only if 1 plays a strategy on the locus of points bordering both the y and x regions in figure 1. Nash equilibrium: $(2\alpha A + \alpha B + (1-3\alpha)C, \frac{1}{2}x + \frac{1}{2}y)$ for $\alpha \in [\frac{1}{4}, \frac{1}{3}]$.

b. Are any of Alistair's strategies (pure or mixed) strictly dominated? If yes, indicate all strategies that are strictly dominated and state which strategy dominates each one. If no, prove it.

From figure 1, if Braxton plays $\frac{1}{2}x + \frac{1}{2}y$, all of Alastair's pure and mixed strategies are best responses, so Alastair has no strictly dominated strategies.

c. Are any of Braxton's strategies (pure or mixed) strictly dominated? If yes, indicate all strategies that are strictly dominated and state which strategy dominates each one. If no, prove it.

From figure 1, if Alastair plays $\frac{1}{2}A + \frac{1}{4}B + \frac{1}{4}C$, all of Braxton's pure and mixed strategies are best responses, so Braxton has no strictly dominated strategies.

Problem 2 (15 points) Carla and Devendra play the following simultaneous move game:

Devendra

Carla	L	R
T	$5,1$	$1,2$
M	$4,4$	$4,2$
B	$2,5$	$5,0$

a. Determine the set of Carla's rationalizable strategies.

See figure 2. Carla's rationalizable strategies are T, M, B, all mixtures of T and M, and M and B.

b. For each of Carla's strategies that is *not* rationalizable, prove that the strategy is strictly dominated by some other strategy.

Carla's non-rationalizable strategies are all mixtures of T and B, as well as all mixtures of T, M, and B. It suffices to show that all mixtures of T and B are strictly dominated.

For $\alpha \in (\frac{1}{4}, \frac{2}{3})$, it is straightforward that M dominates $\alpha T + (1-\alpha)B$. Now, for $\alpha \leq \frac{1}{4}$, we show that the strategy $2\alpha M + (1-2\alpha)B)$ strictly dominates $\alpha T + (1-\alpha)B$. Consider figure 3. The proposed strategy dominates $\alpha T + (1-\alpha)B$ both when 2 plays L and when 2 plays R, so it must also be superior for all mixtures of L and R.

Finally, for $\alpha \geq \frac{2}{3}$, consider the alternative strategy $(2\alpha - 1)T + 2(1-\alpha)M$. Look at figure 4. The alternative strategy gives a payoff superior to $+(1-\alpha)B$ against all possible mixed strategies for player 2, and so $+(1-\alpha)B$ is a strictly dominated strategy.

Problem 3 (15 points) Consider the normal form game $G(r)$:

Felicity

Elnora	I	N
I	r, r	$r-1, 0$
N	$0, r-1$	$0, 0$

In this game, strategy I represents investing, and strategy N represents not investing. Investing yields a payoff of r or $r-1$, according to whether the player's opponent invests or not. Not investing yields a certain payoff of 0.

Describe the set of Nash equilibria of $G(r)$ for each $r \in [-2, 3]$.

First, if $r \in (1, 3]$, strategy I is dominant for both players, and so r, r is the unique Nash equibrium. Likewise, if $r \in [-2, 0)$, N is strictly dominant for each player, and so N, N is the unique Nash equilibrium. If $r \in [0, 1]$, then both (I, I), (N, N), and $((1-r)I + rN, (1-r)I + rN)$ are Nash equilibria, with the last being a mixed equilibrium in which each player plays I with probability r.

Problem 4 (15 points) Gage and Hattie compete in a race. At the start of the race, both players are 6 steps away from the finish line. Who gets the first turn is determined by a toss of a fair coin; the players then alternate turns, with the results of all previous turns being observed before the current turn occurs.

During a turn, a player chooses from these four options:

- Do nothing at cost 0;
- Advance 1 step at cost 2;
- Advance 2 steps at cost 7;
- Advance 3 steps of at cost 15.

The race ends when the first player crosses the finish line. The winner of the race receives a payoff of 20, while the loser gets nothing. Assume there is no discounting, but that all else equal each player prefers to finish the game more quickly.

a. Find all subgame perfect Nash equilibria of this game. (Hint: In all SPNE, a player's choice at a decision node only depends on the number of steps he has left and on the number of steps his opponent has left. To help take advantage of this you might want to write down a table. How many possible states of the world are there? Solve for what one player will do at each possible state. Is the game symmetric?)

Which move is optimal for player G depends only on n_G and n_H, where n_i is the number of steps that remain for player i. Assume that it is currently Gage's turn, and solve for his optimal move choice for all 36 possible values of n_G and n_H. See table 6. The explanation below reasons through the derivation of the 36 optimal strategies for both players in as careful a way as possible. Significant points will be awarded for reasoning that the first mover must necessarily move one space at a time towards the finish.

(I) If $n_G = 1$, Gage's optimal move is 1, regardless of n_H.

(II) If $n_G \geq 2$ and $n_H = 1$, if G does nothing, then by (I), H will finish in the next round. If $n_G \in \{2, 3\}$, then it is profitable for G to finish the race immediately, and so he does so by taking n_G steps. If $n_G \in \{4, 5, 6\}$, G cannot finish the race this turn; since he knows that H will finish next round, G does nothing.

(III) If $n_G = 2$ and $n_H \in \{2, 3\}$, then if G advances one step, by (II) H will finish in the next round. Given this, the best G can do is to finish immediately, so he takes 2 steps.

(IV) If $n_G = 3$ and $n_H = 2$, if G advances 1 or 2 steps, (II) and (III) imply that H will finish in the next round. Given this, G finishes immediately.

(V) Suppose that $n_G = n_H = 3$. If G advances 1 or 2 steps, (II) and (IV) imply that H will finish in the next round. Given this, G finishes immediately.

(VI) Suppose $n_G = 2$ and $n_H \geq 4$. Then, if G takes one step, (II) implies that he will win next round. Since taking 1 step twice costs less than taking 2 steps once, G takes on step.

(VII) Suppose $n_G \geq 4$ and $n_H = 2$. Then, regardless of what G does, H will eventually win, so G does nothing.

(VIII) Suppose $n_G = 3$ and $n_H \geq 4$. Then, G takes 1 step.

(IX) Suppose $n_G \geq 4$ and $n_H = 3$, then G does nothing (as in (VII)).

(X) Suppose $n_i = 4$ and $n_j \geq 4$. Then the cheapest way for G to win is to take 1 step 4 times. In fact, if G takes 1 step now, (IX) and other earlier statements imply that he will win in precisely this fashion. Therefore, G takes 1 step.

(XI) Suppose $n_G \geq 5$ and $n_H = 4$. Then, (X), (VIII), and (VI) tell us that G will win if and only if he advances to at least $n_G = 3$ during this turn. If $n_G = 5$, advancing 2 steps immediately and then winning the race is the only way G can earn a positive payoff, so he takes 2 steps. If $n_G = 6$, then he cannot earn a positive payoff by taking 3 steps now (since $15 + 2 + 2 + 2 = 21 > 20$), so he does nothing.

(XII) Suppose $n_G = n_H = 5$. Then G advances 2 steps (as in (XI)).

(XIII) If $n_G = 6$ and $n_H = 5$, then G does nothing (as in (XI)).

(XIV) If $n_G = 5$ and $n_H = 6$, then G advances 1 step (as in (X)).

(XV) If $n_G = n_H = 6$, then G advances 1 step (as in (X)).

Table 6 summarizes how many steps G will take for each of the 36 states of the world. Since the game is symmetric, the SPNE outcome is for the first mover to move one step at a time towards the finish line, and the second mover to never move at all.

b. Suppose that Gage wins the coin toss. Compare his equilibrium behavior with his optimal behavior in the absence of competition. Provide intuition for any similarities or differences you find.

We have just seen that the player who wins the coin toss takes one step at a time and eventually wins the race. This is exactly what this player would do in the absence of competition. The point here is that by the backward induction argument, the player who loses the coin toss realizes that any effort he makes will be in vain, and so he doesnt make any effort at all. Realizing this, the other player can act as if his opponent did not exist.

Problem 5 (15 points) Senator Iacovelli and Senator Jacoban are bargaining over which policy should be implemented out of the set $\{X, Y, Z\}$. The game they play is as follows:

- First, Sen. Iacovelli vetoes one of the three policies
- Second, after observing Sen. Iacovelli's choice, Sen. Jacoban vetoes one of the remaining policies
- The policy that has not been vetoed at this point is implemented

Sen. Iacovelli's preferences are $X \succ Y \succ Z$, while Sen. Jacoban's are given by $Z \succ Y \succ X$.

a. Solve for the game's subgame perfect Nash equilibria. Be precise.

See figure 5. I vetoes Z in the first round, and in the second round, J vetoes Y, X, and X, respectively, at his three information sets. The SPNE outcome is that policy Y is implemented.

b. Now suppose the game is changed so that Sen. Jacoban moves first, followed by Sen. Iacovelli. Solve for the modified game's subgame perfect Nash equilibrium. Be precise.

Similar reasoning applies: by backward induction, J vetoes X in the first round, and I vetoes Z, Z, and Y at his three information sets, respectively, and the equilibrium outcome is that policy Y is implemented.

Problem 6 (15 points) Kirt and Lila are engaged in a joint project. If person $i \in \{K, L\}$ invests effort $x_i \in [0, 1]$ in the project, at cost $c(x_i)$, the outcome of the project is worth $f(x_K, x_L)$. The worth of the project is split equally by Kirt and Lila, regardless of their effort levels, so that each gets a payoff of $\frac{1}{2}f(x_K, x_L) - c(x_i)$. Suppose effort levels are chosen simultaneously.

a. Suppose $f(x_K, x_L) = 3x_K x_L$ and that $c(x_i) = x_i^2$. Find the Nash equilibrium effort levels of this simultaneous move game.

K solves the following, taking x_L as given: $\max_{x_K} \frac{3}{2}x_K x_L - x_K^2$, which has a maximum located at $x_K^* = \frac{3}{4}x_L$. Similarly, L's payoff is maximized at $x_L^* = \frac{3}{4}x_L^*$. The only values that satisfy both of these equations are $(x_K^*, x_L^*) = (0, 0)$. Therefore, the unique Nash equilibrium of this game is located at $(0, 0)$.

b. Is there a pair of effort levels that yield higher payoffs for both players than do the Nash equilibrium effort levels in part a.?

In the Nash equilibrium described in part a, both players get a payoff of 0. Now consider the alternative arrangement $x_K = x_L = 1$. Here, each player gets a payoff of $\frac{1}{2}$, and is therefore better off than in the Nash equilibrium.

$BR_1 : \Sigma_2 \Rightarrow \Sigma_1$ $\quad$ $BR_2 : \Sigma_1 \Rightarrow \Sigma_2$

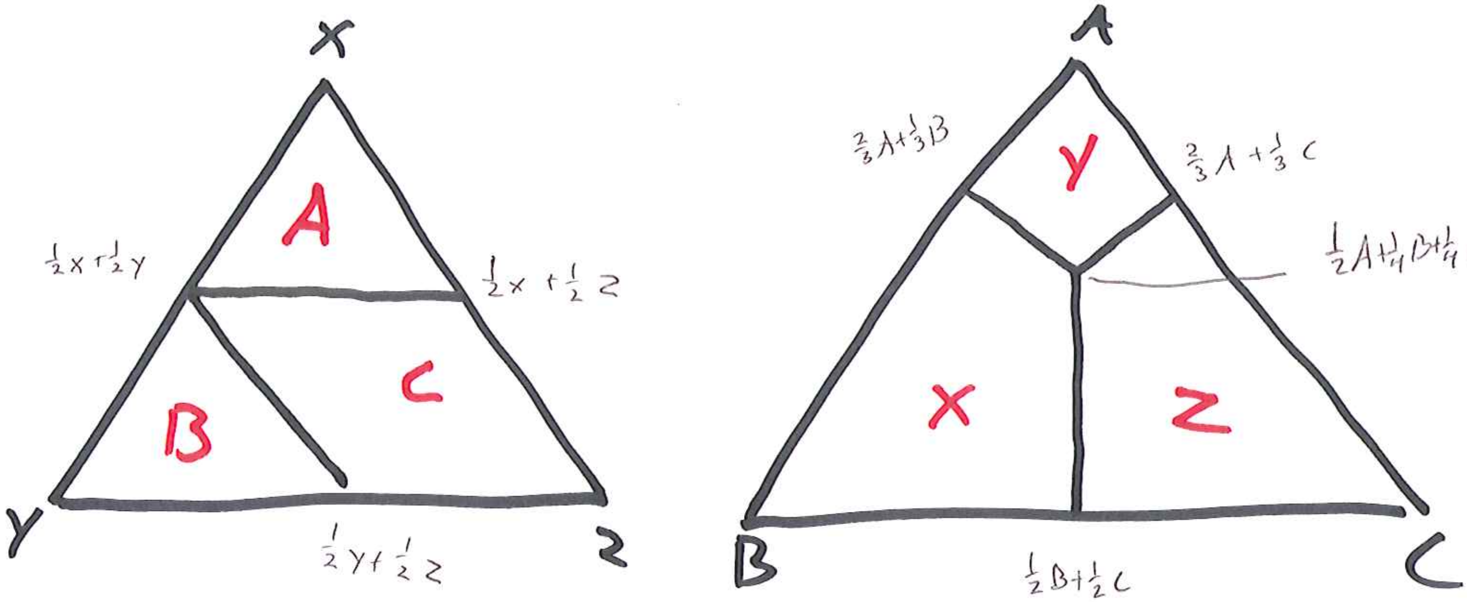

figure 1

$BR_1 : \Sigma_2 \Rightarrow \Sigma_1$

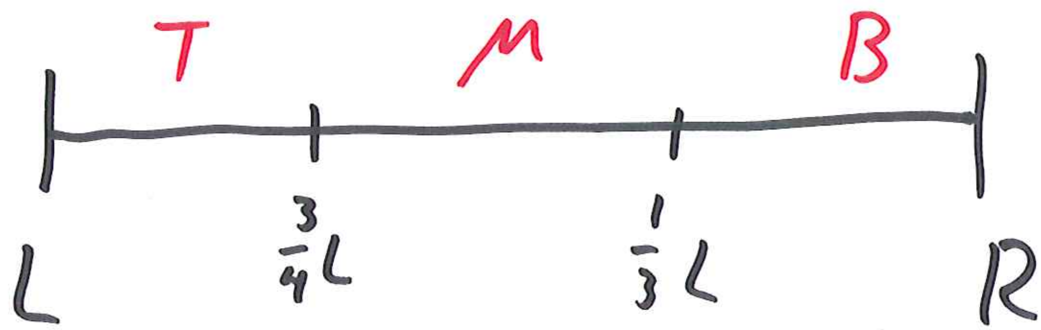

figure 2

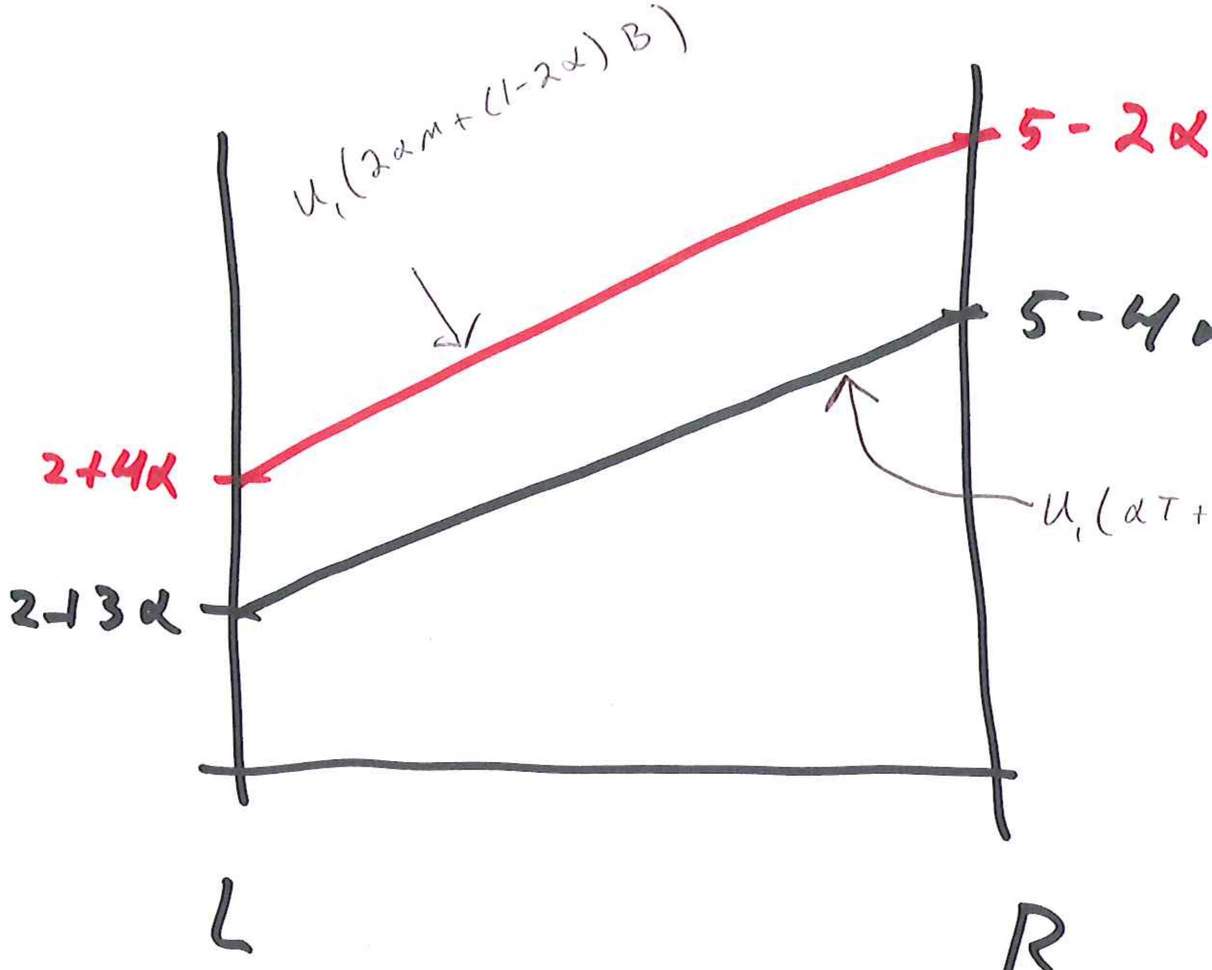

figure 3 $\alpha \leq \frac{1}{4}$

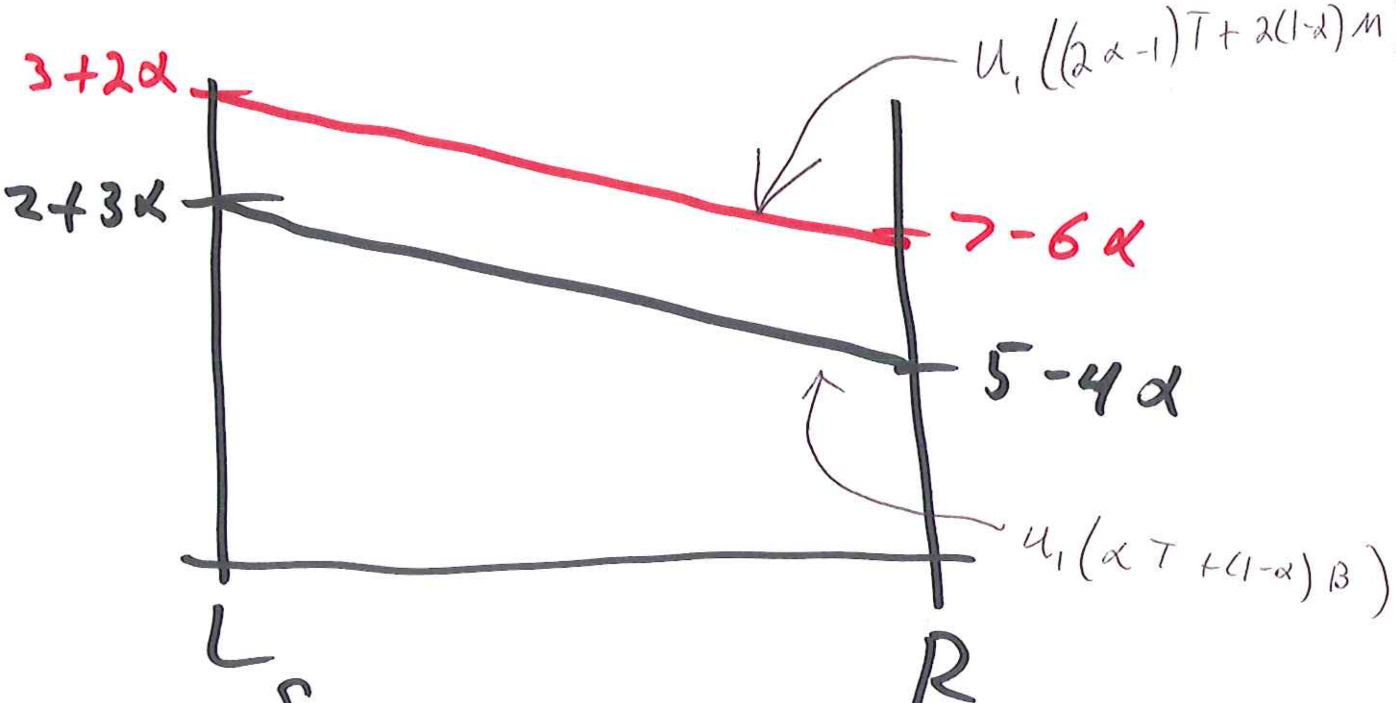

figure 4 $\alpha \geq \frac{2}{3}$

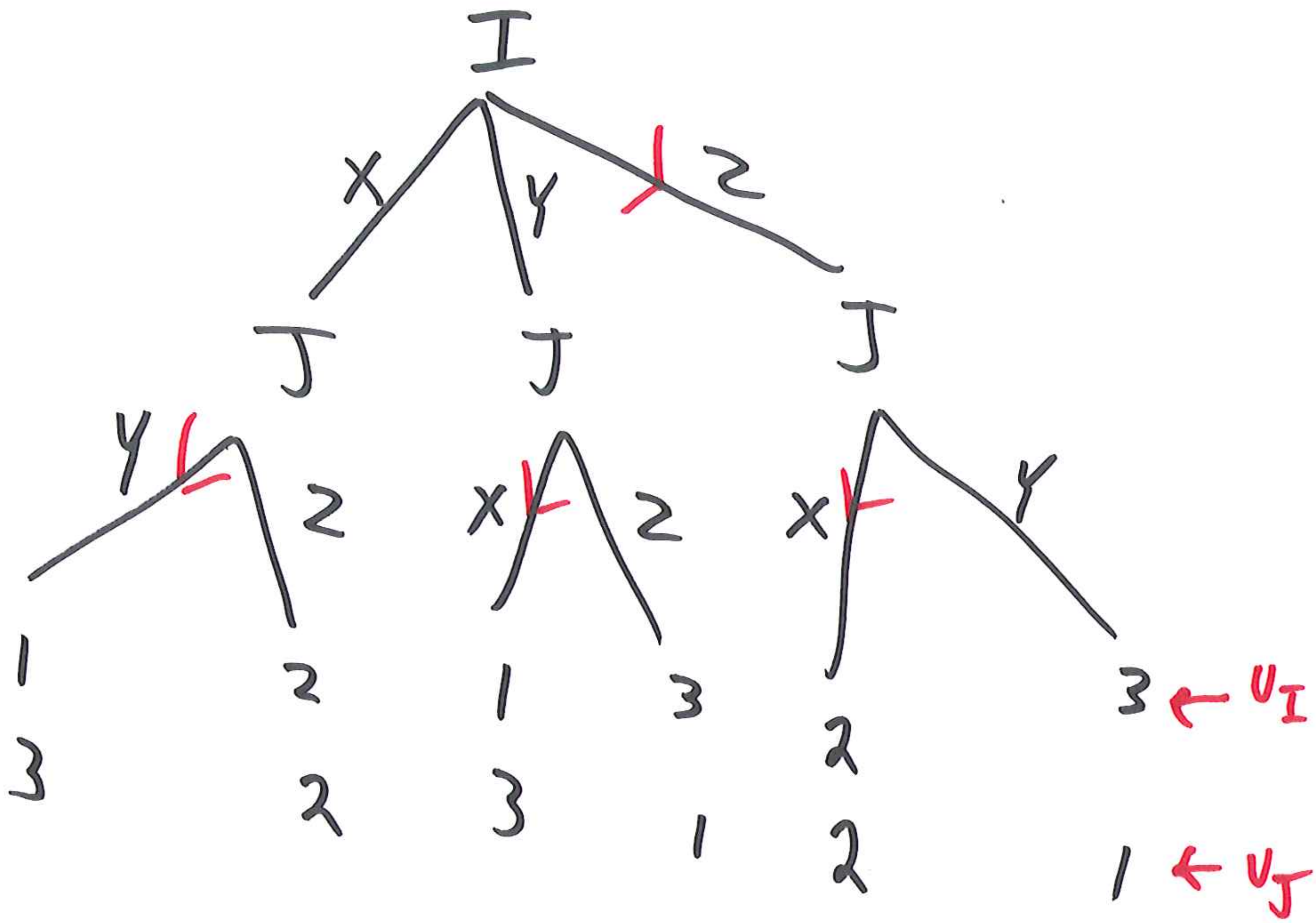

figure 5

Steps left for Hattie

Steps left for Gage	6	5	4	3	2	1
6	1,1	0,1	0,1	0,1	0,1	0,1
5	1,0	2,2	2,1	0,1	0,1	0,1
4	1,0	1,2	1,1	0,1	0,1	0,1
3	1,0	1,0	1,0	3,3	3,2	3,1
2	1,0	1,0	1,0	2,3	2,2	2,1
1	1,0	1,0	1,0	1,3	1,2	1,1

Table 6

Red numbers indicate optimal strategy for G+H for each of 36 states.