Unit 4.3: Uncertainty

Michael Malcolm

June 18, 2011

Up until now, we have been considering consumer choice problems where the consumer chooses over outcomes that are known. However, many choices in economics involve uncertainty. When you choose to buy a stock instead of putting your money in the bank, the outcome is uncertain at the point that you buy the stock. The reason you buy car insurance is because, at the point you buy the insurance, you don't know whether you'll have an accident.

1 Expected Value

Consider a situation involving uncertainty with n possible outcomes. We will call the outcomes $\{x_1, ..., x_n\}$. The probability associated with outcome x_1 is p_1, etc... Then the *expected value* is.

$$EV = \sum_{i=1}^{n} x_i p_i$$

For example, consider tossing a 6-sided die and winning the amount shown on the face of the die. Making a list of the outcomes and associated probabilities:

Outcome	Probability
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

Then the expected value of this game is the probability weighted sum of the outcomes.

$$\begin{aligned} EV &= \sum_{i=1}^{n} x_i p_i \\ &= \left(\frac{1}{6}\right)1 + \left(\frac{1}{6}\right)2 + \left(\frac{1}{6}\right)3 + \left(\frac{1}{6}\right)4 + \left(\frac{1}{6}\right)5 + \left(\frac{1}{6}\right)6 = 3.5 \end{aligned}$$

Notice that, if you play this game once, you never win 3.5. You win 1, 2, 3, 4, 5 or 6. Think of 3.5 as the long-run average value per game if you played many times.

2 Expected Utility

Consumers assign utility to various levels of wealth. Typically, these utility functions obey *diminishing marginal utility of wealth.* What this means is that the increase in utility from a given amount of wealth is lower when a consumer starts off with more wealth. This is a reasonable assumption – $1000 of additional wealth is going to raise a poor person's utility by more than it raises a rich person's utility. Graphically, this means that the utility function over various levels of wealth is a concave function. Utility rises as wealth rises, but at a slowing rate.

Solving problems involving uncertainty typically just involves comparing the expected utility that the consumer will receive from various options. For example, consider a consumer whose utility function over wealth is $U(W) = \sqrt{W}$. He has a house worth $400,000, but there is a 1% chance that his house will be destroyed in a fire, in which case it will be worth nothing.

Making a table to describe the possible outcomes:

Wealth	Probability
400,000	0.99
0	0.01

In this case, his expected wealth is:

$$EW = 0.99(400,000) + 0.01(0) = 396,000$$

Utility from any level of wealth W is $U(W) = \sqrt{W}$. So the expected *utility* from the consumer's housing wealth is:

$$EU = 0.99\sqrt{400,000} + 0.01\sqrt{0} = 626.131$$

Now, suppose that the consumer can buy insurance for $5000 that will cover the value of his house in the event of a loss. In this case, his wealth is $395,000 for sure since he has to pay the cost of the insurance regardless of whether he has a fire. Thus, his utility is $U(W) = \sqrt{395,000} = 628.49$.

Notice what is going on here. His expected (average) wealth is higher if he just tolerates the risk – expected wealth of $396,000 without the insurance but certain wealth of $395,000 with the insurance. However, his *utility* is higher by buying the insurance – he has utility of 628.49 in this case, but expected utility of only 626.13 if he is not insured. Since average utility is higher with the insurance, he buys the insurance for his house.

3 A Risky Investment

Here is another example. An individual with $200,000 of wealth has an opportunity to undertake a risky investment for $50,000. With probability p, the investment is successful and the investor doubles the money he put in. With probability $1-p$ the investment is not successful and the investor loses all the money he put in. Suppose that his utility function over his wealth is $U(W) = \sqrt{W}$. What is the lowest success probability p for which he should make the investment?

Again, solving these problems just involves comparing utility from the various options. By not undertaking the investment, the individual's wealth is $200,000 for sure. Upon undertaking the investment, the individual's wealth is $250,000 with probability p when the investment is successful, but only $150,000 with probability $1 - p$ when the investment is not successful.

Expected utility from undertaking the investment is $EU = p\sqrt{250,000} + (1 - p)\sqrt{150,000}$. Utility from not undertaking the investment is $\sqrt{200,000}$. Thus, the individual will make the investment as long as:

$$
\begin{aligned}
p\sqrt{250,000} + (1-p)\sqrt{150,000} &\geq \sqrt{200,000} \\
500p + 387.298(1-p) &\geq 447.214 \\
112.702p &\geq 59.916 \\
p &\geq 0.532
\end{aligned}
$$

4 Risk Aversion

Consider a consumer whose utility over various levels of wealth is as follows:

Wealth	Utility
\$10	70
\$26	105
\$40	120
\$70	140

This utility function is pictured in figure 1. Notice that this utility function reflects diminishing marginal utility of wealth. Increasing wealth from \$10 to \$40 raises utility by 50 units, but increasing wealth from \$40 to \$70 raises utility by only 20 units. This accounts for the concave shape.

Now, suppose that the consumer has an opportunity to take on a bet that will pay \$10 with probability 0.5 and will pay \$70 with probability 0.5.

His expected wealth from this bet is

$$EW = 0.5(10) + 0.5(70) = 40$$

His expected *utility* from this bet is

$$EU = 0.5(70) + 0.5(140) = 105$$

On average, the consumer ends up with \$40 from this gamble and with expected utility of 105. However, notice that if we just gave the consumer \$40 for sure, then his utility would have been 120. That is, the consumer would rather have \$40 with certainty than a gamble that averages out to \$40. Graphically, the expected utility of the gamble is located on the midpoint of the chord that connects the two possible outcomes, as shown in figure 1(if the gamble probabilities had not been 50/50, then the expected utility from the gamble would have been located closer to one or the other outcome on the chord, depending upon which probability was higher). Observe that the average utility from the gamble, shown on the chord, is lower than the utility if the consumer simply had \$40 – the expected value of the gamble – for sure.

The intuition for this is fairly straightforward. Because of diminishing marginal utility of wealth, the *decline* in utility when the consumer loses the gamble is greater than the *increase* in utility when he wins. Thus, his utility is higher if he has \$40 for sure rather than a gamble that averages out to \$40. We say that this consumer is *risk averse*.

Now, notice from the table that wealth of \$26 for sure gives the consumer the same utility as the gamble. \$26 is called the *certainty equivalent* – the amount of wealth without any uncertainty that would yield the same utility as the gamble. The certainty equivalent is shown in figure 1.

Expected wealth from the gamble is \$40, but the consumer would have had the same utility by having wealth of \$26 for certain. This difference of \$14 is known as the *risk premium*. Intuitively, it represents the highest amount of money that he would give up in order to avoid the gamble altogether. The risk premium is also shown in figure 1.

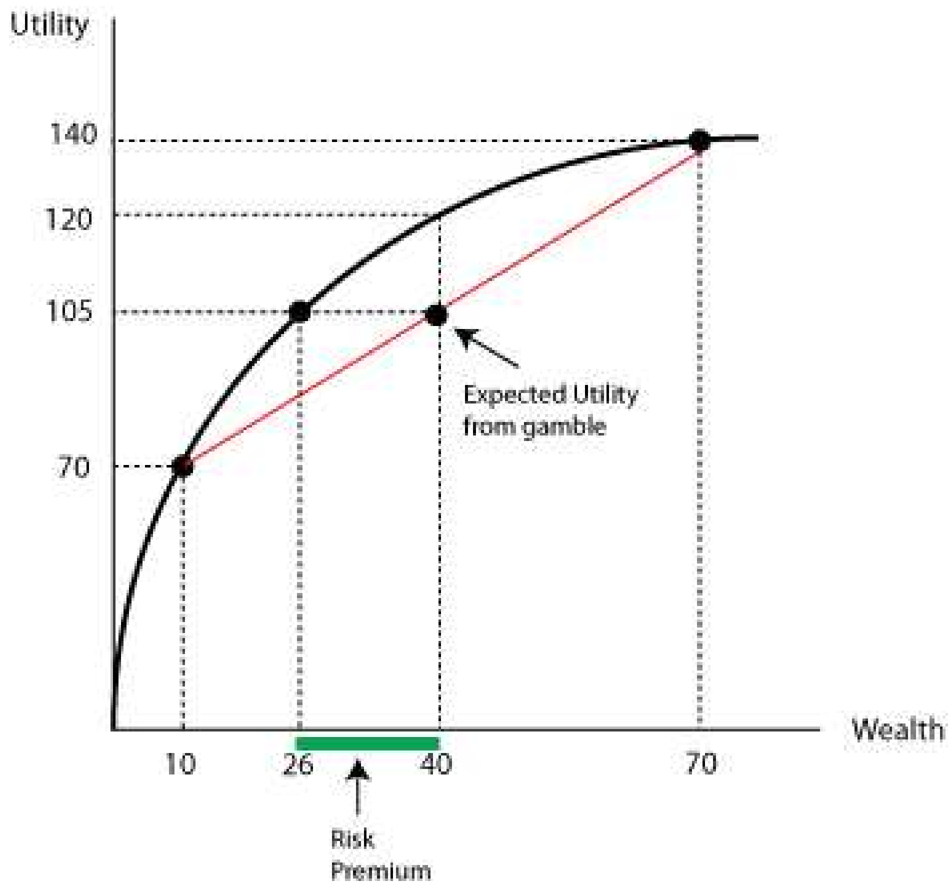

Figure 1: Risk averse consumer

5 Risk Aversion and Insurance

The kind of logic above is exactly why insurance companies operate. Consumers are willing to accept a lower expected wealth in order to avoid uncertainty. In the fire example, from earlier in the section, we saw that the consumer was willing to pay a $5000 insurance premium and tolerate a lower expected wealth in order to avoid the risk of a big loss.

Consider a consumer with utility function $U(W) = \sqrt{W}$. He faces the following uncertainty.

Wealth	Probability
$144	$\frac{2}{3}$
$225	$\frac{1}{3}$

The expected value of his wealth is:

$$EW = \frac{2}{3}(144) + \frac{1}{3}(225) = 171$$

His expected utility is:

$$EU = \frac{2}{3}\sqrt{144} + \frac{1}{3}\sqrt{225} = 13$$

Now, suppose that an insurance company offers to fully resolve the uncertainty and simply give the consumer wealth of $171. After he buys the insurance, he faces no uncertainty. However, he has to pay a premium of P in order to buy the insurance. What is the highest premium P that he would be willing to pay?

Again, solving this problem is just a matter of comparing the utility from the two options. As shown above, if the consumer accepts the gamble, then his expected utility is 13. By taking the insurance, the consumer has a guaranteed wealth of 171, minus the premium of P, for utility of $\sqrt{171 - P}$. Thus, the consumer buys the insurance as long as:

$$\begin{aligned} \sqrt{171 - P} &\geq 13 \\ 171 - P &\geq 169 \\ P &\leq 2 \end{aligned}$$

To review this example using the terminology that we developed earlier, see figure 2. The gamble gives the consumer an expected utility of 13. Notice that he could have had this same utility by having wealth of $169 for certain since $U(W) = \sqrt{169} = 13$ in this case. In other words $169 is the certainty equivalent. Since the gamble is worth an average of $171, but the consumer would have accepted $169 for sure to generate the same utility, we say that the risk premium is $2. Intuitively, this is exactly the same as the maximum willingness to pay for full insurance that we calculated above.

6 Different Attitudes Towards Risk

Up until now, we have been considering the case of a *risk averse* consumer. As shown in figure 3, the consumer's expected utility from a gamble over W_1 and W_2 is shown as $EU(gamble)$ on the chord connecting the two. This is lower than the consumer's utility if he would have received the average value of the gamble EW with no uncertainty – which is shown on the diagram as $U(EW)$.

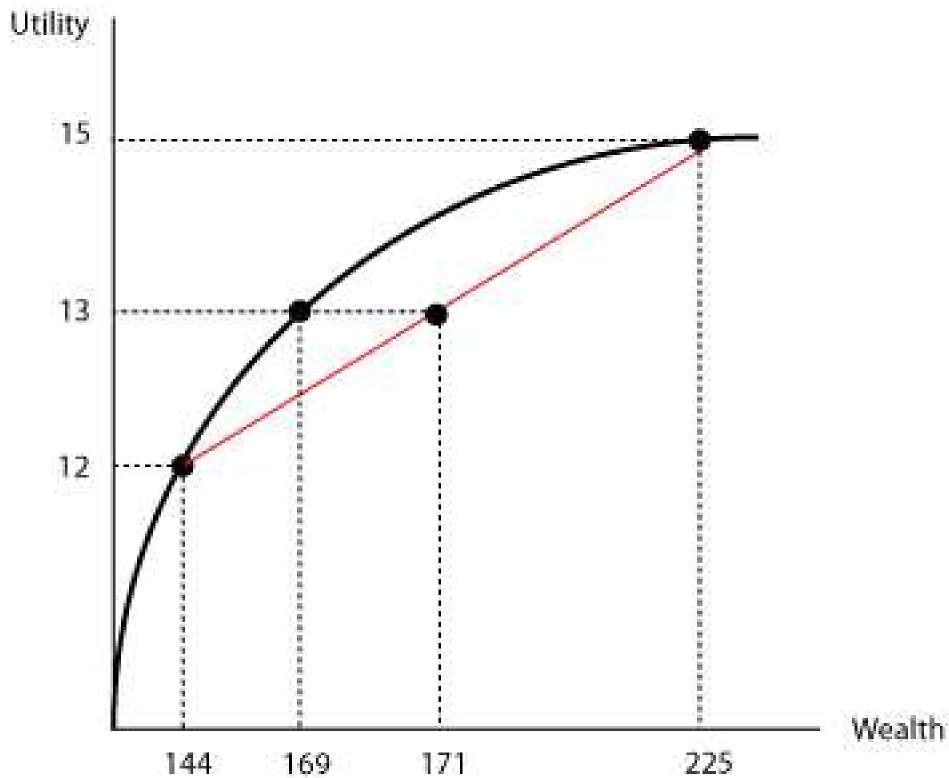

Figure 2: Risk aversion and insurance

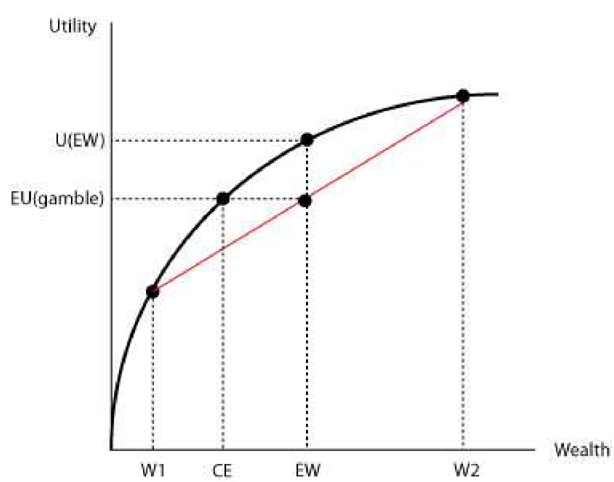

Figure 3: Risk averse consumer

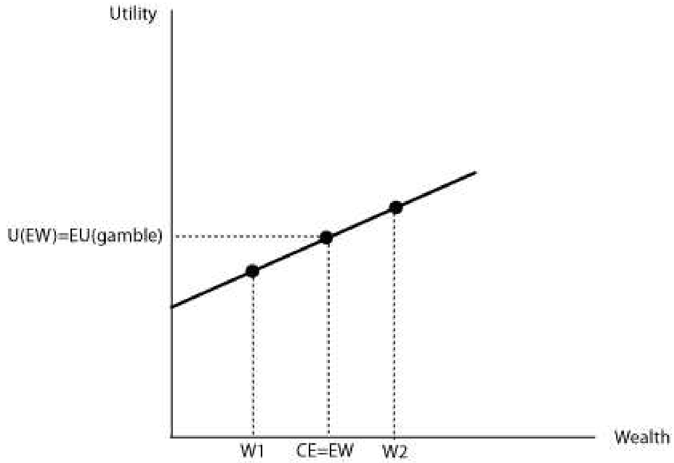

Figure 4: Risk neutral consumer

Rather than take the gamble, the consumer would have accepted wealth equal to the certainty equivalent CE, since this gives him the same utility as the gamble. The fact that CE is lower than the expected value of the gamble reflects the consumer's risk aversion and willingness to pay to avoid the gamble.

The consumer in figure 4 is *risk neutral*. A gamble over W_1 and W_2 gives exactly the same utility as receiving the expected value of the gamble for sure. Risk neutrality is a convenient assumption in many settings because it allows us to simply compare the expected values of two alternatives to make a decision. Since a risk neutral consumer has no like or dislike of risk, all he cares about is the expected value of two options. For example, suppose that investment A pays $1000 with probability 0.1 and $0 with probability 0.9. Investment B pays $150 with probability 0.5 and $50 with probability 0.5. A risk neutral consumer is indifferent over these two investments since both have an expected value of $100. In contrast, a risk averse consumer clearly would have preferred investment B since it is far less risky.

The consumer in figure 5 is *risk loving*. A gamble over W_1 and W_2 generates higher utility $EU(gamble)$ than the utility of receiving the expected value of the gamble for sure $U(EW)$. In this case, the certainty equivalent CE is actually higher than the expected value of the gamble EW. In other words, this consumer actually enjoys risk, so we would have to pay him money to *not* gamble. Contrast with a risk averse consumer, who was willing to give up money in order to avoid a gamble.

7 Arrow-Pratt Risk Aversion

Concave utility functions represent risk averse preferences. In fact, it is easy to see from the diagram that the more concave the utility function is, the more risk averse the consumer is. The more concave the utility function becomes, the lower is the expected utility from gambling versus the utility from receiving the expected wealth for sure. This means that the consumer is more risk averse and willing to pay more to avoid the risk.

Mathematically, the concavity of a function is measured by its second derivative. This leads to the *Arrow-Pratt measure of risk aversion*, designated ρ.

$$\rho = -\frac{U''(W)}{U'(W)}$$

For the utility function that we have been working with $U(W) = \sqrt{W}$, the first and second derivatives are, respectively, $U'(W) = \frac{1}{2}W^{-\frac{1}{2}}$ and $U''(W) = -\frac{1}{4}W^{-\frac{3}{2}}$. Then the Arrow-Pratt risk aversion is:

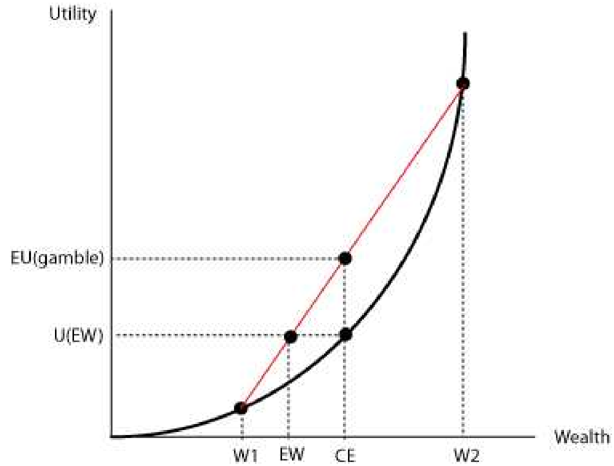

Figure 5: Risk loving consumer

$$
\begin{aligned}
\rho &= -\frac{U''(W)}{U'(W)} \\
&= -\frac{-\frac{1}{4}W^{-\frac{3}{2}}}{\frac{1}{2}W^{-\frac{1}{2}}} \\
&= \frac{1}{2}W^{-1} \\
&= \frac{1}{2W}
\end{aligned}
$$

Notice that, as W rises, risk aversion falls for this utility function. This makes good sense. People with very little wealth will tend to be more risk averse over the same gamble than people with a lot of wealth.

8 Different Risk Preferences over Different Levels of Wealth

Why would someone who gambles buy insurance for his house at the same time? Gambling suggests risk loving behavior, but insuring your house against a fire suggests risk aversion. The difference is that gambling typically involves a small amount of welath, but potentially losing your house in a fire represents a gamble over a very large amount of wealth. Many consumers are risk-loving over small gambles but risk-averse over larger gambles. See figure 6, which illustrates the utility function of such a consumer.

9 Fair Insurance

Insurance is called *actuarially fair* if the insurance premium is equal to the company's expected payout to the customer. In the fire example from the beginning of this section, the insurance company paid out the value of the $400,000 house with probability 0.01 when there was a fire but paid out $0 with probability 0.99 the rest of the time. The insurance company's expected payout to this customer is $4000, so that is the actuarially fair premium in this case.

A risk averse person will fully insure, given access to actuarially fair insurance. Actually, as shown previously, the company can charge more than the actuarially fair premium, reflecting the risk premium that

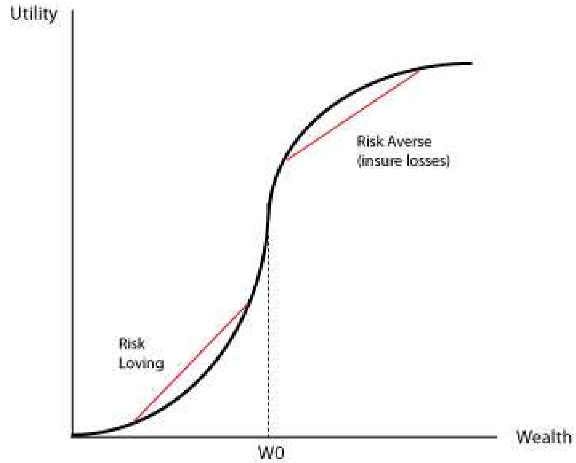

Figure 6: Risk loving over small amounts of wealth and risk averse over larger amounts of wealth

a risk averse consumer is willing to pay to avoid uncertainty (though this might be difficult in a competitive market with many insurance companies competing for customers).

10 Idiosyncratic and Systemic Risk

Insurance basically allows the holder to transfer risk to the insurance company. Why would an insurance company agree to do this? The key is that, with a large number of policies, the law of averages works in the insurance company's favor. In any given year, about the same proportion of customers will experience a house fire. The insurance company doesn't face a tremendous amount of uncertainty in it's total payout, whereas a single customer faces a significant uncertainty over a very large portion of his wealth.

The catch is that this argument only applies to *idiosyncratic* risks like house fires, which basically occur independently across customers. This is in contrast to *systemic* risks like floods, which are correlated – many customers tend to make claims at the same time. Insurance companies are, in fact, much less likely to insure systemic risks than idiosyncratic risks. An insurance company offering fire insurance doesn't face much uncertainty because the risk is idiosyncratic and unlikely to happen to many customers in the same year; the company can expect a few claims in any given year with near certainty. An insurance company offering flood insurance faces a different situation; most likely, there is no flood and the company pays out nothing. However, there is some probability that a flood will occur, in which case the insurance company ends up paying out a huge amount of money since many customers will suffer losses. The expected payout may be the same in both cases, but the flood insurance is much riskier for the insurance company.

Insurance companies are risk averse, too, and are much less likely to insure against floods than they are to insure against fires.

11 Investment Strategy

The discussion of idiosyncratic and systemic risk from above tells us something about investment strategy. A risky financial portfolio is one where all the assets are highly correlated – they go up and down together. To reduce risk, investors should diversify and invest in assets that are going to increase in value when other assets fall in value and vice versa. For example, someone who owns a large amount of stock in BMW might reduce his exposure to risk by buying some stock in Mercedes – if BMW goes bankrupt, we would expect a corresponding increase in profits for Mercedes (however, it is difficult to diversify against more systemic risk like a recession, which would lower the profits of both companies).