Economics 590, fall 2007
Instructor: Jeremy Sandford

Homework 2
due 9/11/2007

Problem 1 (Images, preimages). *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = 3x + 2y$ for $(x, y) \in \mathbb{R}^2$.*

a. Find the image under f of the set $[0, 5] \times [3, 7]$, or $f([0, 5] \times [3, 7])$.

b. Find the preimage under f of the set $\{10\}$, that is $f^{-1}(\{10\})$.

c. Is f injective? Surjective? Prove your claims.

Problem 2 (Convexity). *True or false: the empty set ϕ is convex. Prove your claim.*

Problem 3 (Cartesian products). *(Sundaram, page 69 #30) Given two subsets A and B of $\mathbb{R}$, recall that their Cartesian product $A \times B \subset \mathbb{R}^2$ is defined as*

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Give an example of a set $X \subset \mathbb{R}^2$ that cannot be expressed as the Cartesian product of sets $A, B \subset \mathbb{R}$.

Problem 4 (Metrics I). *For $x, y \in \mathbb{R}^n$, consider the distance function given by*

$$\begin{aligned} d(x, y) &= 1 \text{ if } x \neq y \\ &\quad\ 0 \text{ if } x = y \end{aligned}$$

Prove that $d(\cdot)$ is a metric on $\mathbb{R}^n$.

Problem 5 (Metrics II). *(Rudin, page 44, #11) For $x \in \mathbb{R}$ and $y \in \mathbb{R}$, define*

$$\begin{aligned} d_1(x, y) &= (x - y)^2 \\ d_2(x, y) &= \sqrt{|x - y|} \\ d_3(x, y) &= |x^2 - y^2| \\ d_4(x, y) &= |x - 2y| \\ d_5(x, y) &= \frac{|x - y|}{1 + |x - y|} \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

Problem 6 (Metrics III). *Is every metric on $\mathbb{R}^2$ order-preserving? That is, if $d_1 : \mathbb{R}^2 \to \mathbb{R}$ and $d_2 : \mathbb{R}^2 \to \mathbb{R}$ are both valid metrics on $\mathbb{R}^2$, is it necessarily the case that, for any $x, y, z \in \mathbb{R}^2$,*

$$d_1(x, y) < d_1(x, z) \Rightarrow d_2(x, y) < d_2(x, z)?$$

Problem 7 (Metrics IV). *Google Maps provides a distance in miles between two addresses. Indeed, the program allows you to input locations in terms of latitude and longitude, so Google Maps could reasonably be said to map any two points in Lexington to a real number representing the distance between those two points. Ignoring technical problems with the website, could the Google Maps algorithm be said to define a metric over the set of all points in Lexington?*