Economics 590, fall 2007

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Homework 5
due 10/9/2007

Problem 1 (Continuous functions).

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases}$$

Show, using the definition of continuity given in class, that f *is continuous at* $\frac{1}{2}$ *but discontinuous at every other point in its domain.*

Problem 2 (Derivatives). *Using the definition of the derivative, find the derivative of the following functions, all mapping* $\mathbb{R}$ *to* $\mathbb{R}$ *(you already know what all these derivatives are; the goal is not to simply write them down, but to prove, using the definition, that they are what you think they are):*

a) $f(x) = x^3$

b) $f(x) = x^n$*, for any integer* n

c) $f(x) = \frac{sin(x)}{x}$

d) $f(x) = ax^n + b$*, for any integer* n *and* $a, b \in \mathbb{R}$

e) $f(x) = c$*, for any* $c \in \mathbb{R}$

Problem 3 (Derivatives II). *Suppose* f *and* g *are defined on* $[a, b]$ *and are differentiable at point* $x \in [a, b]$*. Find, using the definition (see admonition in problem 2), each of the following:*

a) $h'(x)$*, where* $h(x) = f(x) + g(x)$

b) $q'(x)$*, where* $q(x) = f(x) * g(x)$

c) $w'(x)$*, where* $w(x) = \frac{f(x)}{g(x)}$

Problem 4 (Maximization). *Let* f *be defined on* $[a, b]$*; show that if* f *has a local maximum at a point* $x \in (a, b)$*, and if* $f'(x)$ *exists, then* $f'(x) = 0$*.*

Problem 5 (Derivatives III). *A consumer has utility function* $u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$*, for some nonzero constant* $\rho < 1$*. He faces prices* p_1 *and* p_2 *and has income* I*.*

a) Solve for his demand functions, $x_1(p_1, p_2, I)$ *and* $x_2(p_1, p_2, I)$*.*

b) Write down a function X *mapping* $\mathbb{R}^3$ *to* $\mathbb{R}^2$ *which describes the consumer's demand system.*

c) Find the Jacobian of the demand system, i.e. the derivative of X*.*

d) Set $\rho = -1$*. Write down an expression for locally approximating* X *around the point* $(p_1, p_2, I) = (9, 25, 2700)$ *using the Jacobian from c).*

e) What value for X *do you get using your approximation from d) at point* $(p_1, p_2, I) = (10, 24, 2650)$*? How close is it to the actual value of the function* X *at this point?*

Problem 6 (Derivatives IV). *A beet farm has production function* $f(r, k, l) = 8r^{\frac{1}{4}}k^{\frac{3}{4}}l$*, where* k *is the farm's quantity of capital,* l *its quantity of labor, and* r *its quantity of land, in acres.*

a) Suppose the farm currently employs 10,000 acres of land, 6,561 units of capital, and 500 workers. Calculate the gradient of the production function at its current production level.

b) Now calculate the gradient of the production function for any values of the inputs (r, k, l).

c) From any initial point (r, k, l) give the direction with the maximal rate of increase in beet production. Make sure to normalize your direction vector to have length one.

d) Suppose again the farm is currently producing at $(r, k, l) = (10000, 6561, 500)$. Solve for how quickly beet production would increase were the farm to add capital and labor in a 3:1 ratio, holding fixed the quantity of land.

e) You are hired by the farm as an advisor and asked what the best ratio would be to add capital and labor, in terms of maximizing the rate of increase in beet production. Only a small increase is possible, and it is not feasible for the farm to add more land. What do you advise?