Economics 590, fall 2007
Instructor: Jeremy Sandford

Midterm
10/23/2007

Directions: **Answer 4 of the 5 equally weighted questions.** Take great pains to support your answers as thoroughly as you are able. Unsupported arguments and unaccompanied pictures are unlikely to receive many points. Remember, the standard you aspire to is having your answers be clear and intelligible to an intelligent person who is not in our class.

You may use results from lectures or books, but be as specific as possible in citing them. You have until 1:55pm. Good luck!

Problem 1 (25 points). *Consider the production function $f(k,l) = 12k^{\frac{1}{3}}l^{\frac{2}{3}}$. That is, a firm employing l labor and k capital produces $f(k,l)$ units.*

a. From an initial position of (k,l), give the ratio in which k and l should be increased to increase $f(k,l)$ most rapidly.

The gradient of f is given by

$$\nabla f(k,l) = \begin{pmatrix} 4k^{-\frac{2}{3}}l^{\frac{2}{3}} \\ \\ 8k^{\frac{1}{3}}l^{-\frac{1}{3}} \end{pmatrix}$$

and so the ratio $k : l$ of maximal increase is $4k^{-\frac{2}{3}}l^{\frac{2}{3}} : 8k^{\frac{1}{3}}l^{-\frac{1}{3}}$, or $l : 2k$.

b. From an initial point of $l = 216$ and $k = 27$, calculate the derivative of f in the direction of maximal increase. Show that, from this initial point, increasing k and l together in this direction increases f more rapidly than does increasing k or l alone.

Using the answer to a, we want to increase k and l in a $4 : 1$ ratio, or in direction $(\frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}})$. We have that $\nabla f(27,216) = \begin{pmatrix} 16 \\ \\ 4 \end{pmatrix}$, and so $D_v f(27,216) = 16 * \frac{4}{\sqrt{17}} + 4 * \frac{1}{\sqrt{17}} = \frac{68}{\sqrt{17}}$, for $v = (\frac{4}{\sqrt{5}}, \frac{1}{\sqrt{5}})$. $\frac{68}{\sqrt{17}}$ is greater than either $\frac{\partial f}{\partial k}$ or $\frac{\partial f}{\partial l}$ evaluated at this point.

c. From an initial position of $(k,l) = (27,216)$, give the direction v in which $D_v f(k,l) = 0$, that is, in which production does not change at all.

$D_v f(27,216) = 0 \iff 16v_1 + 4v_2 = 0$, for $v_1^2 + v_2^2 = 1$. Thus, $v_1 = -\sqrt{1 - v_2^2}$, and so we need $-16\sqrt{1 - v_2^2} + 4v_2 = 0$, or $v_2 = \sqrt{\frac{16}{17}}$ and thus $v_1 = -\sqrt{\frac{1}{17}}$. Note that $v_1 = \sqrt{\frac{1}{17}}$ and $v_2 = -\sqrt{\frac{16}{17}}$ work also.

d. Provide an economic interpretation of your answer in c.

The direction v from part c is the direction in which the isoquant line through $(k,l) = (27,216)$ moves. The locus of points $t * v$, for $t \in \mathbb{R}$, would trace out a line tangent to the isoquant curve at $(k,l) = (27,216)$.

Problem 2 (25 points). *In $\mathbb{R}$ with metric $d(x,y)$, a sequence $\{x_n\}$ is called a Sandford sequence if, for any $\epsilon > 0$, there exists a number $N(\epsilon)$ such that $n,m > N(\epsilon)$ implies that $d(x_n,x_m) < \epsilon$.*

Prove that any convergent sequence in $\mathbb{R}$ is a Sandford sequence.

If a sequence x_n converges to a point x, than, for any $\delta > 0$ there exists $N_\delta > 0$ such that $n > N$ implies that $d(x_n, x) < \delta$. Pick $\delta = \frac{\epsilon}{2}$, and note that, for $n, m > N_\delta$, $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, with the first inequality being the triangle inequality.

Problem 3 (25 points). *If A is a non-empty closed subset of $\mathbb{R}^2$, and $x \notin A$, is there a point in A that is nearest to x, under the standard metric $d(y, z) = \sqrt{y^2 + z^2}$? Is it unique? Prove your claims.*

If A is bounded, d is clearly continuous and thus obtains a minimum on a closed and bounded set, and we are done. If A is not bounded, pick any point $y \in A$, and then define $B = \{z \in \mathbb{R}^2 : d(z, x) \leq d(y, x)\}$ to be the set of all points in $\mathbb{R}^2$ within distance $d(x, y)$ of point x, i.e. at least as close to x as y is. The closest point to x in A is surely in B, as y is in both A and B. Now, B is simply a circle containing its boundary, and so is closed. Intersections of closed sets are closed, and so $A \cap B$ is closed, and is clearly bounded. $d(x, y)$ is continuous in y and thus obtains a minimum in y on $A \cap B$ by the Weierstrass theorem, and so there is a point in A which is closest to x. The point need not be unique. Suppose $A = \{(1, 0)\} \cup \{(-1, 0)\}$, and $x = (0, 0)$. Both points in A are then equidistant from x.

Problem 4 (25 points). *Let $g : \mathbb{R} \to \mathbb{R}$ be a function (not necessarily continuous) which has a maximum and minimum on $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Does $h(x) = f(g(x))$ necessarily have a maximum on $\mathbb{R}$? Prove your answer, or provide a detailed counterexample.*

No. Consider

$$g(x) = \begin{cases} -1, & \text{if } x < -1 \\ x, & \text{if } x \in (-1, 0) \\ -1, & \text{if } x \in [0, 1) \\ 1, & \text{if } x > 1 \end{cases}$$

and $f(x) = -x^2$. Then, $h(x) = f(g(x))$ has no maximum. To see this, $h(x) = -1$ for all x outside of the interval $(-1, 0)$, and $h(x) > -1$ for all $x \in (-1, 0)$. But, for any $x \in (-1, 0)$, $h(\frac{1}{2}x) > h(x)$, and so no maxmimum can exist in this interval.

Problem 5 (25 points). *Prove whether or not each of the following functions is quasiconcave:*

a. $f : \mathbb{R}_+ \to \mathbb{R}$, $f(x) = -x^2$

$f''(x) = -2 < 0$, and so f is concave, and thus quasiconcave.

b. $f : \mathbb{R}_+ \to \mathbb{R}$, $f(x) = x^2$

f is increasing and thus quasiconcave.

c. $f : [0, 2\pi] \to \mathbb{R}$, $f(x) = sin(x)$

$\{x \in [0, 2\pi] : f(x) \geq 0\} = [0, \pi] \cup \{2\pi\}$, which is not a convex set, and so f is not quasiconcave.

d. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = 4x_1 + 2x_2 - x_1^2 + x_1x_2 - x_2^2$

f has Hessian $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$, which is negative definite; f is therefore concave and thus quasiconcave.