Final exam

Answer each of the following 6 equally-weighted questions. Carefully state the appropriate definitions and theorems, where applicable, and argue how they apply. Make sure that every step in your argument follows logically and directly from the immediately preceding one. For questions 3, 5, and 6, be sure to prove that either that your answers are global maximizers/minimizers over the constraint set or that no maximizer/minimizer exists.

Problem 1 Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be a convex function satisfying $f(0) = 0$.

a. Show that for all $k \geq 1$ we have $kf(x) \leq f(kx)$.

b. If, instead, $k \in [0, 1)$, what is the relationship between $kf(x)$ and $f(kx)$?

Problem 2 Consider the problem of choosing consumption and wealth sequences to maximize

$$\sum_{t=0}^{\infty} \beta^t \left(log(c_t) + \gamma \log(c_{t-1}) \right) \text{ subject to } c_t + w_{t+1} = Aw_t^{\alpha}$$

where $\beta \in (0, 1)$, $\gamma > 0$, $\alpha \in (0, 1)$, and w_0 and c_{-1} are given. Here, c_t denotes consumption at time t and w_t is wealth at the beginning of time t. Note that consumption purchases generate a stream of utility that extends for more than one period.

a. Write the Bellman equation associated with the maximization problem.

b. Guess that the value function has the form $V[w, c_{-1}] = a + b \log(w) + d \log(c_{-1})$; verify your guess and go as far as you can in solving for the coefficients a, b, d.

c. Give expressions for optimal choices of consumption c and one-period-ahead wealth w', as functions of state variables w and c_{-1}.

Problem 3 Solve the following minimization problem.

$$\min\ x_1 + x_2 \qquad \text{subject to} \qquad \begin{aligned} (x_1 - 1)^2 + x_2^2 - 1 &= 0 \\ (x_1 - 3)^2 + x_2^2 - 4 &= 0 \end{aligned}$$

Problem 4 Let $f : (1, \infty) \to (1, \infty)$ be given by

$$f(x) = \frac{1}{2}\left(x + \frac{2}{x} \right)$$

a. Show that f is a contraction (under the standard Euclidean metric).

b. Show that $(1, \infty)$ is NOT complete.

c. Does f have a fixed point? If so, is it unique? Prove your answer.

Problem 5 Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be functions given by

$$\begin{aligned} f(x, y) &= -7x^3 + 4x^2y - 3xy^2 + 9y^3 \\ g(x, y) &= x^3 - y^2 \end{aligned}$$

Solve the problem of maximizing f subject to the constraint $g \geq 0$.

Problem 6 Find the max and min of the function $f(x) = 6x^3 - 14x^2 + 10x$ over the interval $[\frac{1}{2}, 2]$.