Homework 3

due 9/22/08

Problem 1 (Closed sets) Consider $\mathbb{R}^n$ with the Euclidean metric. Prove that $F \subset \mathbb{R}^n$ is closed if and only if the complement of F is open.

Suppose F is closed. Choose $x \in F^c$. Then x is not a limit point of F, so there is some $\epsilon > 0$ such that $B(x, \epsilon) \bigcap F$ is empty, and thus $B(x, \epsilon) \subset F^c$. That this is true for any $x \in F^c$ implies f^c is open.

Now, suppose F^c is open. This means that for any $x \in F^c$, there is some $\epsilon > 0$ such that $B(x, \epsilon) \subset F^c$, which, in turn, means that $F \bigcap B(x, \epsilon)$ is empty. Therefore, x is not a limit point of F. That F has no limit points in F^c implies F is closed.

Problem 2 (Intersections of open sets) In $\mathbb{R}^n$ with the Euclidean metric,

a. Prove that the intersection of any finite number of open sets is an open set.

For $\{O_i\}_{i=1}^n$, pick $x \in \bigcap_{i=1}^n O_i$. That $x \in O_1 \Rightarrow B(x, \epsilon_1) \subset O_1$ for some $\epsilon_1 > 0$. That $x \in O_2 \Rightarrow B(x, \epsilon_2) \subset O_2$ for some $\epsilon_2 > 0$. And so on, giving us $\epsilon_1, \epsilon_2, ..., \epsilon_n$. Set $\epsilon = \min\{\epsilon_1, \epsilon_2, ..., \epsilon_n\}$. It is clear that $B(x, \epsilon) \subset O_i$ for each i, and thus $B(x, \epsilon) \subset \bigcap_{i=1}^n O_i$, so $\bigcap_{i=1}^n O_i$ is open, QED.

b. Give an example of an infinite (countable or uncountable) collection of open sets such that the intersection is not open. Be as explicit as possible.

This will work for sets $\{A_n\}$ which get 'smaller' in the sense that $A_n \subset A_{n-1}$ for all n, such that the intersection $\bigcap A_n$ is non-empty. Consider for example in $\mathbb{R}$ $A_n = (1 - \frac{1}{n}, 3 + \frac{1}{n})$, which has intersection $\cap A_n = [1, 3]$ (a relatively simple proof will establish this).

c. Give an example of an infinite (countable or uncountable) collection of closed sets such that the union is not closed. Again, be very explicit about what the union is and how you know it is not closed.

This will work for sets $\{A_n\}$ which get 'larger' in the sense that $A_n \supset A_{n-1}$ for all n, such that the union $\bigcup A_n \neq \mathbb{R}^n$. Consider for example in $\mathbb{R}$ $A_n = [1 - \frac{n}{n+1}, 3 + \frac{n}{n+1}]$, which has intersection $\bigcap A_n = [1, 3]$.

Problem 3 (Closed sets II) Consider $\mathbb{R}^n$ with the Euclidean metric. Prove the following statement:

$F \subset \mathbb{R}^n$ is closed if and only if for every sequence $\{x_n\}$ contained in F,

$$\lim_{n \to \infty} x_n = x \quad \Rightarrow \quad x \in F. \tag{1}$$

Again, use the definition of 'closed' given in class.

First, if every sequence in F converges to a point in F, consider a point $x \in F^c$. If x were a limit point of F, then for every $n = 1, 2, ...$, there would exist a point $y_n \in F$ such that $y_n \in B(x, \frac{1}{n})$. But then this would describe a sequence y_n which converges to $x \in F^c$, an impossibility. Conclude that x is not a limit point of F for all $x \in F^c$, and thus that F is closed.

Now suppose that F is closed. We show that every sequence contained in F must converge to a point in F. Suppose this were not the case. Then, there would exist a $\{x_n\}$ such that $x_n \to x \in F^c$, which implies that x is a limit point of F, as for any $\epsilon > 0$, $B(x, \epsilon)$ contains terms from $\{x_n\}$, which are in F. But this is a contradiction, as F is closed. Conclude that any sequence in F converges to a point in F.

Problem 4 (Extreme values) (Sundaram page 68, #16) Find the supremum, infimum, maximum, and minimum, if they exist, for the following sets:

a. $A_1 = \{x \in [0,1] : x \text{ is irrational}\}$$\sup A_1 = 1, \inf A_1 = 0, \max A_1, \min A_1$ do not exist.

b. $A_2 = \{x : x = \frac{1}{n}, \text{ for } n = 1, 2, ...\}$$\sup A_2 = \max A_2 = 1, \inf A_2 = 0, \min A_2$ does not exist.

c. $A_3 = \{x : x = 1 - \frac{a}{n}, \text{ for } n = 1, 2, ...\}$ (note: take a to be some real number in $(0,1)$)$\sup A_3 = 1, \max A_3$ does not exist, $\min A_3 = \inf A_3 = 1 - a$

d. $A_4 = \{x \in [0, \pi] : sin(x) > \frac{1}{2}\}$$\sup A_4 = \frac{5\pi}{6}, \inf A_4 = \frac{\pi}{6}$, $\max A_4, \min A_4$ do not exist.

e. $A_5 = \phi$, the empty set (Take $\phi \subset \mathbb{R}$ here. Then every element of $\mathbb{R}$ is both an upper and a lower bound of ϕ, and so there is clearly no least upper bound or greatest lower bound. The max and min do not exist either.

Problem 5 (Convex sets) A set A is convex if, for every $x_1, x_2 \in A$, and for every $\alpha \in (0,1)$,

$$\alpha x_1 + (1 - \alpha) x_2 \in A \tag{2}$$

(this is the usual definition from class).

If a set B satisfies $\frac{1}{2}x_1 + \frac{1}{2}x_2 \in B$ for all $x_1, x_2 \in B$, does it follow that B is convex?

No. Consider the set $\mathbb{Q} \subset \mathbb{R}$. For any $q_1, q_2 \in \mathbb{Q}$, $\frac{1}{2}q_1 + \frac{1}{2}q_2 \in \mathbb{Q}$, but, for example, $\frac{1}{\pi}q_1 + (1 - \frac{1}{\pi})q_2 \notin \mathbb{Q}$, and so $\mathbb{Q}$ is not convex.

Problem 6 (Metric spaces and open sets) Fact: sets are open and closed *relative to the metric space in which they are contained.* That is, a set which is open in metric space $(A, d_1(\cdot))$ may not be open when seen as a subset of $(B, d_2(\cdot))$.

Demonstrate your understanding of this by arguing that the set $(0,5)$ is open when seen as a subset of $(\mathbb{R}, |\cdot|)$, but not open as a subset of $(\mathbb{R}^2, |\cdot|)$, where $|\cdot|$ is the standard Euclidean metric.

In $\mathbb{R}$, $B(x, \epsilon) = (x - \epsilon, x + \epsilon)$. In $\mathbb{R}^2$, $B(x, \epsilon) = \{(y_1, y_2) \in \mathbb{R}^2 : \sqrt{(y_1 - x_2)^2 + (y_2 - x_2)^2}$. Clearly, the interval $(0,5)$ is open in $\mathbb{R}$. To see it is not open in $\mathbb{R}^2$, pick point $(2,0)$ and note that $B((2,0), \epsilon) = \{(y_1, y_2) : \sqrt{(y_1 - 2)^2 + y_2^2} < \epsilon\}$, a circle containing, for example, the point $(2, \frac{1}{2}\epsilon)$, which is not in the segment $(0,5)$ of the horizontal axis.

Problem 7 (Rational numbers) $\mathbb{Q} = \{x \in \mathbb{R} : x = \frac{a}{b}, \text{ for integers } a, b\}$ denotes the set of rational numbers. Is $\mathbb{Q}$ an open subset of the Euclidean space $(\mathbb{R}, |\cdot|)$, where $|\cdot|$ is the Euclidean metric? Is $\mathbb{Q}$ a closed subset of the same?

$\mathbb{Q}$ is not open in $\mathbb{R}$, as $B(1, \epsilon) = (1 - \epsilon, 1 + \epsilon)$, which contains $\frac{1}{\pi} + (1 - \frac{1}{\pi})(1 + \epsilon)$, an irrational number.

Problem 8 (Open covers) An *open cover* of a set $A \subset \mathbb{R}^n$ is a collection of open sets $\{O_i\}_{i \in I}$, $O_i \subset \mathbb{R}^n$ for each $i \in I$, such that $A \subset \cup_{i \in I} O_1$.

a. Go as far as you can in proving that every open cover of the interval $[0,1] \subset \mathbb{R}$ has a finite subcover, that is that for any sets $\{O_i\}_{i \in I}$ such that $[0,1] \subset \cup_{i \in I} O_i$, there exist n elements of $\{O_i\}_{i \in I}$, call them $O_{i(1)}, O_{i(2)}, ..., O_{i(n)}$, such that $[0,1] \subset \cup_{j=1}^n O_{i(j)}$.

b. Give an example of an open cover of $(0,1)$ which has no finite subcover. Consider the collection of open sets centered at $\frac{1}{n}$ with radius $\frac{4}{5}(\frac{1}{n} - \frac{1}{n+1}) = \frac{4}{5}\frac{1}{n(n+1)}$, for $n = 1, 2, 3, ...$ I leave it as an exercise to verify that these sets cover all of $(0,1)$ and that there is no finite subcover. For the latter, observe that the points $\frac{1}{n}$ each fall into exactly one set, so removing any of these sets would destroy the cover. As there are countably many such sets, there is necessarily no finite subcover. Note that the point 0 is not covered by this collection. Were we required to cover 0, the set containing 0 would also contain infinitely many of the points $\frac{1}{n}$, allowing us to "discard" all but a finite number of sets from the above collection and still cover $(0,1)$.