Homework 4

due 9/29/08

Problem 1 (Weierstrass theorem) (Sundaram, page 98 #13) A monopolist faces a downward sloping inverse-demand curve $p(x)$ that satisfies $p(0) < \infty$ and $p(x) \geq 0$ for all $x \in \mathbb{R}_+$. The cost of producing x units is given by $c(x) \geq 0$, where $c(0) = 0$. Suppose $p(\cdot)$ and $c(\cdot)$ are both continuous on $\mathbb{R}_+$. The monopolist wishes to maximize profit, $\pi(x) = xp(x) - c(x)$, subject to the constraint $x \geq 0$.

a) Suppose there is $x^* > 0$ such that $p(x^*) = 0$. Show that the Weierstrass theorem can be used to prove the existence of a solution to this problem.

b) Now suppose instead there is $\tilde{x} > 0$ such that $c(x) \geq xp(x)$ for all $x \geq \tilde{x}$. Show, once again, that the Weierstrass theorem can be used to prove existence of a solution.

c) What about the case where $p(x) = \bar{p}$ for all x (the demand curve is infinitely elastic) and $c(x) \to \infty$ as $x \to \infty$?

Problem 2 (Weierstrass theorem II) (Sundaram, page 97 #2) Suppose $A \subset \mathbb{R}^n$ is a set consisting of a finite number of points $\{x_1, x_2, ..., x_p\}$. Show that any function $f : A \to \mathbb{R}$ has a maximum and a minimum on A. Is this result implied by the Weierstrass theorem? Explain.

Problem 3 (Weierstrass theorem III) (Sundaram, page 97 #1) Prove or counter the following statement:

If f is a continuous real-valued function on a bounded (but not necessarily closed) set A, then $\sup f(A)$ is finite. (nb. $\sup f(A) = \sup\{y \in \mathbb{R} : y = f(x) \text{ for some } x \in A\}$).

Problem 4 (Sequences) (Sundaram, page 67 #3) Let $\{x_n\}$, $\{y_n\}$ be sequences in $\mathbb{R}^n$ such that $x_n \to x$ and $y_n \to y$. For each n, let $z_n = x_n + y_n$, and let $w_n = x_n * y_n$. Show that $z_n \to (x + y)$ and $w_n \to x * y$.

Problem 5 (Sequences II) In $\mathbb{R}^n$ with metric $d(x, y)$, a sequence $\{x_n\}$ is called a *Cauchy sequence* if, for any $\epsilon > 0$, there exists a number $N(\epsilon)$ such that $n, m > N(\epsilon)$ implies that $d(x_n, x_m) < \epsilon$.

Prove that any convergent sequence in $\mathbb{R}^n$ is a Cauchy sequence.

Problem 6 (Metric spaces) Prove or counter each of the following statements:

a. If A is a non-empty closed subset of $\mathbb{R}^n$, and $x \notin A$, there is a point in A that is nearest to x, under the metric $d(x, y) = \sqrt{\Sigma_{i=1}^n (x_i - y_i)^2}$.

b. If A is a non-empty open subset of $\mathbb{R}^n$, and $x \notin A$, there is a point in A that is nearest to x, under the metric $d(x, y) = \sqrt{\Sigma_{i=1}^n (x_i - y_i)^2}$.

c. If A is a non-empty closed and bounded subset of $\mathbb{R}^n$, and $x \notin A$, there is a unique point in A that is nearest to x, under the metric $d(x, y) = \sqrt{\Sigma_{i=1}^n (x_i - y_i)^2}$.

Problem 7 (Basic optimization) Prove or counter the following statement:

If $g : \mathbb{R} \to \mathbb{R}$ is a function (not necessarily continuous) which has a maximum and minimum on $\mathbb{R}$, and if $f : \mathbb{R} \to \mathbb{R}$ is continuous, $h(x) = f(g(x))$ necessarily has a maximum on $\mathbb{R}$.