Homework 5

due 10/6/08

Problem 1 (Continuous functions) (Sundaram, page 72)

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if x is irrational} \end{cases}$$

Show that f is continuous at $\frac{1}{2}$ but discontinuous at every other point in its domain.

Problem 2 (Sequences) Show that no unbounded sequence $\{x_n\} \subset \mathbb{R}$ converges to a point $p \in \mathbb{R}$

Problem 3 (Derivatives)

a. Find the derivative of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ at any point $x \in (-\infty, 0) \cup (0, \infty)$, and show that the function is not differentiable at 0.

b. Show that the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x|x|$ is differentiable for all $x \in \mathbb{R}$. What is the derivative?

Problem 4 (Continuity and inverse images) (Sundaram, page 71)

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Show that the set

$$\{x \in \mathbb{R}^n : f(x) = 0\}$$

is a closed set.

Problem 5 ($\liminf$, $\limsup$) (Sundaram page 68) Find the $\limsup$ and the $\liminf$ of each of the following sequences:

a. $x_n = (-1)^n$, $n = 1, 2, 3, ...$

b. $x_n = (-1)^n + \frac{1}{n}$, $n = 1, 2, 3, ...$

c. $\{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, ...\}$

d. $x_n = 1$ is n is odd, and $x_n = -\frac{n}{2}$ if n is even

Problem 6 (Derivatives II) Find the derivative of each of the following functions with domain and codomain $\mathbb{R}$, from the definition of derivative[1]:

a. $f(x) = 2x^3$

b. $f(x) = 12x^{-2}$

b. $f(x) = 3x + 2$

Problem 7 (Taylor expansions I)

a. Approximate the function $f(x) = e^x$ around $x = 0$ with separate Taylor expansions of degrees 1,2, and 3 (you may use without proof the fact that $f^{(n)}(0) = 1$ for all n). Call these $g_1(x)$, $g_2(x)$, and $g_3(x)$.

b. Calculate the interval over which $g_i(x)$ is no more than 10% away from $f(x)$, that is, in which $\frac{|g_i(x) - f(x)|}{|f(x)|} \leq .1$, for $i = 1, 2, 3$ (you can approximate this with the help of a computer if necessary).

c. Repeat parts a and b for $f(x) = \sqrt{x+1}$. (you need not prove what $f^{(n)}(x)$ is).

[1]The point of this problem is to demonstrate comfort working with the definition. Do not simply write out what the derivative is.

d. Repeat parts a and b for $f(x) = \ln(x)$, this time expanded about $x = 1$ (you need not prove what $f^{(n)}(x)$ is).

Problem 8 (Taylor expansions II)

Suppose you needed to calculate $4.2^{\frac{3}{2}}$ without using a computer. Show that this is possible via a first degree Taylor series expansion of $f(x) = x^{\frac{3}{2}}$ about $x = 4$. How close is your approximation to the actual value?