Homework 5

due 10/6/08

Problem 1 (Continuous functions) (Sundaram, page 72)

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if x is irrational} \end{cases}$$

Show that f is continuous at $\frac{1}{2}$ but discontinuous at every other point in its domain.

For any $\epsilon > 0$,

$$|f(x) - \frac{1}{2}| = \begin{cases} |x - \frac{1}{2}|, & \text{if } x \text{ is rational} \\ |1 - x - \frac{1}{2}| = |\frac{1}{2} - x| = |x - \frac{1}{2}|, & \text{if x is irrational} \end{cases} \quad (1)$$

Set $\delta = \epsilon$. Then, $|x - \frac{1}{2}| < \delta$ implies that $|f(x) - f(\frac{1}{2})| < \epsilon$, as required for continuity at $\frac{1}{2}$. For $x \neq \frac{1}{2}$ rational, consider $\epsilon = |\frac{1}{2} - x|$. Then, for any $\delta > 0$, consider $y = \min\{\frac{1}{\pi}x + (1 - \frac{1}{\pi})\frac{1}{2}, \frac{1}{\pi}x + (1 - \frac{1}{\pi})(x + \delta)\}$. y is clearly irrational. Moreover, if $x < \frac{1}{2}$, $f(y) > \frac{1}{2}$, while if $x > \frac{1}{2}$, $f(y) < \frac{1}{2}$, and thus $|f(y) - f(x)| > \epsilon$. Thus there exists $\epsilon > 0$ such that for all $\delta > 0$ at least one element $y \in (x - \delta, x + \delta)$ such that $|f(y) - f(x)| > \epsilon$, and thus f is not continuous at $x \neq \frac{1}{2}$ rational. A nearly identical proof would work for $x \neq \frac{1}{2}$ irrational.

Problem 2 (Sequences) Show that no unbounded sequence $\{x_n\} \subset \mathbb{R}$ converges to a point $p \in \mathbb{R}$

Suppose not. Then $x_n \to p$, for some $p \in \mathbb{R}$. But that x_n is unbounded implies that for any $M > 0$, $|x_n| > M$ for sufficiently large n. Set M equal to $|p| + \epsilon$ for some $\epsilon > 0$; then $|x_n - p| > \epsilon$ for all sufficiently large n, contradicting our assumption that $x_n \to p$.

Problem 3 (Derivatives)

a. Find the derivative of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ at any point $x \in (-\infty, 0) \cup (0, \infty)$, and show that the function is not differentiable at 0.

b. Show that the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x|x|$ is differentiable for all $x \in \mathbb{R}$. What is the derivative?

Problem 4 (Continuity and inverse images) (Sundaram, page 71)

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Show that the set

$$\{x \in \mathbb{R}^n : f(x) = 0\}$$

is a closed set.

Let $B \subset \mathbb{R}$ be any closed set. Then, B^c is open, and so

$$\begin{aligned} f^{-1}(B^c) &= f^{-1}(\mathbb{R} - B) \\ &= f^{-1}(\mathbb{R}) - f^{-1}(B) \\ &= \mathbb{R}^N - f^{-1}(B) \end{aligned} \quad (2)$$

is open, and thus $f^{-1}(B)$ is closed, as desired (two of these steps are glossed over, try to show them yourselves).

Problem 5 ($\liminf$, $\limsup$) (Sundaram page 68) Find the lim sup and the lim inf of each of the following sequences:

a. $x_n = (-1)^n$, $n = 1, 2, 3, ...$

$\limsup x_n = 1$, $\liminf x_n = -1$

b. $x_n = (-1)^n + \frac{1}{n}$, $n = 1, 2, 3, ...$

$\limsup x_n = 1$, $\liminf x_n = -1$

c. $\{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, ...\}$

$\limsup x_n = \infty$, $\liminf x_n = 1$

d. $x_n = 1$ is n is odd, and $x_n = -\frac{n}{2}$ if n is even

$\limsup x_n = 1$, $\liminf x_n = -\infty$

Problem 6 (Derivatives II) Find the derivative of each of the following functions with domain and codomain $\mathbb{R}$, from the definition of derivative[1]:

a. $f(x) = 2x^3$

The derivative of f at x is m satisfying $\lim_{h\to 0} \frac{r(h)}{h}$, where

$$\begin{aligned} \frac{r(h)}{h} &= \frac{2(x+h)^3 - 2x^3 - mh}{h} \\ &= \frac{2x^3 + 6xh^2 + 6x^2h + 2h^3 - 2x^3 - mh}{h} \\ &= 6xh + 6x^2 + 2h^2 - m \end{aligned} \tag{3}$$

and so $\lim h \to 0\frac{r(h)}{h} = 6x^2 - m$. Conclude that the derivative of f is $f'(x) = 6x^2$.

b. $f(x) = 12x^{-2}$

The derivative of f at x is m satisfying $\lim_{h\to 0} \frac{r(h)}{h}$, where

$$\begin{aligned} \frac{r(h)}{h} &= \frac{12(x+h)^{-2} - 12x^{-2} - mh}{h} \\ &= \frac{\frac{12}{x^2+2xh+h^2} - \frac{12}{x^2} - mh}{h} \\ &= \frac{\frac{-24xh-12h^2}{x^2(x+h)^2} - mh}{h} \\ &= \frac{-24x - 12h}{x^2(x+h)^2} - m \end{aligned} \tag{4}$$

and so $\lim h \to 0\frac{r(h)}{h} = -24x^{-3} - m$. Conclude that the derivative of f is $f'(x) = -24x^{-3}$.

b. $f(x) = 3x + 2$

Problem 7 (Taylor expansions I)

a. Approximate the function $f(x) = e^x$ around $x = 0$ with separate Taylor expansions of degrees 1,2, and 3 (you may use without proof the fact that $f^{(n)}(0) = 0$ for all n). Call these $g_1(x)$, $g_2(x)$, and $g_3(x)$.

b. Calculate the interval over which $g_i(x)$ is no more than 10% away from $f(x)$, that is, in which $\frac{g_i(x)-f(x)}{f(x)} \leq .1$, for $i = 1, 2, 3$ (you can approximate this with the help of a computer if necessary).

c. Repeat parts a and b for $f(x) = \ln(x)$ (you need not prove what $f^{(n)}(x)$ is).

d. Repeat parts a and b for $f(x) = \sqrt{x+1}$ (you need not prove what $f^{(n)}(x)$ is).

Problem 8 (Taylor expansions II)

[1]The point of this problem is to demonstrate comfort working with the definition. Do not simply write out what the derivative is.

Suppose you needed to calculate $4.2^{\frac{3}{2}}$ without using a computer. Show that this is possible via a first degree Taylor series expansion of $f(x) = x^{\frac{3}{2}}$ about $x = 4$. How close is your approximation to the actual value?