Homework 8

due 11/10/08

Problem 1 (Constrained optimization I) (Sundaram, page 168)

Solve the following maximization problem:

$$\begin{aligned} \text{maximize } & \log(x) + \log(y) \\ & \text{subject to } x^2 + y^2 = 1 \\ & x \geq 0, y \geq 0 \end{aligned}$$

The constraint set is closed and bounded and the objective function is continuous. However, the objective function is not defined on $\{x \in \mathbb{R}^2 : x = 0 \text{ or } y = 0\}$, so the Weierstrass theorem cannot be applied directly. Note that the objective function tends to $-\infty$ as $x \to 0$ or $y \to 0$. Thus there is some $\epsilon > 0$ such that the solution to the above problem is the same as the solution to a modified problem with the inequality constraints replaced by $y \geq \epsilon$, $x \geq \epsilon$; this modified problem clearly has a solution by the Weierstrass theorem. Apply the theorem of Lagrange to maximize the objective function over $\mathbb{R}^2 \bigcap \{x \in \mathbb{R}^2 : x^2 + y^2 - 1 = 0\}$. The constraint qualification is met at all $(x, y) \neq (0, 0)$, which is not in the constraint set. Thus, the solution to the problem appears as a point satisfying

$$\begin{pmatrix} \frac{1}{x} \\ \frac{1}{y} \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$
$$x^2 + y^2 = 1$$

for some λ. There is only one such point, $x = y = \frac{1}{\sqrt{2}}$.

Problem 2 (Constrained optimization II) (Sundaram, page 168)

A firm produces two outputs y and z using a single input x. The set of attainable output levels $H(x)$ from an input use of x is given by

$$H(x) = \{(y, z) \in \mathbb{R}^2 | y^2 + z^2 \leq x\}$$

The firm has available to it a maximum of one unit of the input x. Letting p_y and p_z denote the prices of the two outputs, determine the firm's optimal output mix.

We are to choose (x, y, z) to maximize $p_y y + p_z z$ over $H(x)$, subject to $1 \geq x \geq 0$. First, as the objective function is monotonic, the solution to the above problem is the same as that of maximizing the objective function over $\tilde{H}(x) = \{(y, z) \in \mathbb{R}^2 : y^2 + z^2 = x\}$. Second, the monotonicity of the objective function immediately implies that the optimal choice of x is 1. The set $\{(y, z) \in \mathbb{R}^2 : y^2 + z^2 = 1\}$ is closed and bounded, while the objective function is continuous, so a maximum exists by the Weierstrass theorem. Apply the theorem of Lagrange to maximize $p_y y + p_z z$ over $\mathbb{R}^2_{++} \bigcap \{(y, z) \in \mathbb{R}^2 : y^2 + z^2 = 1\}$. The constraint qualification is met everywhere over this set. We check both the points at which the derivative of the objective function is proportional to the derivative of the constraint function $y^2 + x^2 - 1$ and the boundaries of the constraint set. The former are found at points (y, z) satisfying

$$\begin{pmatrix} p_y \\ p_z \end{pmatrix} = \lambda \begin{pmatrix} 2y \\ 2z \end{pmatrix}$$
$$y^2 + z^2 = 1$$

for some λ. There is one such point, $y = \frac{1}{\sqrt{1+\left(\frac{p_z}{p_y}\right)^2}}, z = \frac{1}{\sqrt{1+\left(\frac{p_y}{p_z}\right)^2}}$. There are two points on the boundary of the constraint set, $y = 1, z = 0$ and $y = 0, z = 1$. It is straightforward, though tedious, to verify that the objective function is maximized among these three points at the first point. Conclude that $y = \frac{1}{\sqrt{1+\left(\frac{p_z}{p_y}\right)^2}}, z = \frac{1}{\sqrt{1+\left(\frac{p_y}{p_z}\right)^2}}$ is the solution to the original problem.

Problem 3 (Constrained optimization III) (Sundaram, page 169)

A consumer has income $I > 0$ and faces a price vector $p \in \mathbb{R}^3_{++}$ for the three commodities she consumes. All commodities must be consumed in nonnegative amounts. Moreover, she must consume at least two units of commodity 2, and cannot consume more than one unit of commodity 1. Assuming $I = 4$ and $p = (1, 1, 1)$, calculate the optimal consumption bundle if the utility function is given by $u(x_1, x_2, x_3) = x_1 x_2 x_3$. What if $I = 6$ and $p = (1, 2, 3)$? Let $h_1 = x_1$, $h_2 = x_2 - 2$, $h_3 = x_3$, $h_4 = 1 - x_3$, $h_5 = I - p_1 x_1 - p_2 x_2 - p_3 x_3$. We are asked to maximize $x_1 x_2 x_3$ s.t. $h_i \geq 0$, $i = 1, 2, 3, 4, 5$. First, note that the constraint set is closed and bounded, and the objective function continuous, so a solution to this problem exists by the Weierstrass theorem. Second, note that constraints h_1 and h_3 will not bind at the optimum so long as $2p_2 < I$. If either binds, the objective function is zero, and it is clear that positive utility can be achieved with $x_2 = 2$ and $x_1 > 0$, $x_3 > 0$. Third, note that the derivatives of h_2, h_4, and h_5 are linearly independent given positive prices, so the constraint qualification is satisfied. Applying the Kuhn-Tucker theorem, there exist λ_2, λ_4, and λ_5 such that the following hold at the constrained maximum:

$$\lambda_2 \leq 0, \lambda_2[1 - x_1] = 0, 1 \geq x_1$$

$$\lambda_4 \leq 0, \lambda_4[x_2 - 2] = 0, x_2 \geq 2$$

$$\lambda_5 \leq 0, \lambda_5[I - p_1 x_2 - p_2 x_2 - p_3 x_3] = 0, I \geq p_q x_1 + p_2 x_2 + p_3 x_3$$

$$\begin{pmatrix} x_2 x_3 \\ x_1 x_3 \\ x_1 x_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_5 \begin{pmatrix} -p_1 \\ -p_2 \\ -p_3 \end{pmatrix}$$

This system of equalities and inequalities has two solutions: $x_1 = x_3 = 0, x_2 \in [2, 4], \lambda_2 = \lambda_4 = \lambda_5 = 0$; and $x_1 = 1, x_2 = 2, x_3 = 1, \lambda_2 = 0, \lambda_4 = -1, \lambda_5 = -2$. Clearly, the objective function is larger when evaluated at the latter. Conclude the problem's solution is then $x_1 = x_3 = 1, x_2 = 2$.

Problem 4 (Constrained optimization IV) (Sundaram, page 169)

Let $T \geq 1$ be some finite integer. Solve the following maximization problem:

$$\text{maximize} \sum_{t=1}^{T} \left(\frac{1}{2}\right)^t \sqrt{x_t}$$

$$\text{subject to} \sum_{t=1}^{T} x_t \leq 1$$

$$x_t \geq 0, t = 1, 2, ..., T$$

A solution to the problem clearly exists by the Weierstrass theorem. As the objective function is monotonically increasing in each of its arguments, the first constraint will hold with equality. The inequality constraints will not bind at the optimum (why not?). Thus, apply the theorem of Lagrange to maximize the objective function over $\mathbb{R}^T_{++} \bigcap \{x \in \mathbb{R}^T : \sum_{t=1}^T x_t = 1\}$. The constraint qualification trivially holds, so the optimum will solve the following system for some λ:

$$\begin{pmatrix} \frac{1}{2\sqrt{x_1}} \\ \frac{1}{4\sqrt{x_2}} \\ ... \\ \frac{1}{2^T\sqrt{x_T}} \end{pmatrix} = \lambda \begin{pmatrix} -1 \\ -1 \\ ... \\ -1 \end{pmatrix}$$

which yields $x_t = \frac{1}{4^{t-1}} x_1$, $t = 2, 3, ..., T$. Thus, $\sum_{t=1}^T x_t = 1 \Rightarrow \sum_{t=1}^T \frac{1}{4^{t-1} x_t} = 1$, so that $x_t = \frac{3}{4(1-(\frac{1}{4})^T)}$, $t = 1, 2, ..., T$.

Problem 5 (Constrained optimization V)

Consider the following maximization problem:

$$\text{maximize } \alpha \log(x_1) + (1-\alpha)\log(x_2)$$
$$\text{subject to } p_1 x_1 + p_2 x_2 \leq I, x \geq 0, y \geq 0 \tag{1}$$

p_1, p_2, and I are unknown parameters, and are all strictly positive.

a. Solve the problem by applying the Kuhn-Tucker theorem; be sure to include all three inequality constraints.

b. Solve the problem again by applying the theorem of Lagrange, according to the following outline: first, argue that the first constraint, $p_1x_1 + p_2x_2 \leq I$, must hold with equality at any maximum. Second, apply the theorem of Lagrange to maximize $\alpha \log(x_1) + (1-\alpha)\log(x_2)$ over $\mathbb{R}^2_{++} \bigcap \{(x_1, x_2) \in \mathbb{R}^2 : p_1x_1 + p_2x_2 = I\}$. Third, check the value of the objective function at the "endpoints," $(\frac{I}{p_x}, 0)$ and $(0, \frac{I}{p_y})$. This should give you sufficient justification to claim that you have solved the problem. Argue that your answer is the same as in part a., where you used the Kuhn-Tucker theorem.