Homework 9

due 11/17/08

Problem 1 (Identifying concave functions) (Sundaram, page 198) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = ax^2 + by^2 + 2cxy + d$. For what values of a, b, c and d is f concave? This function has second derivative $D^2 f(x, y) = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$. According to the result on Sundaram page 55, $D^2 f$ is negative semidefinite (and thus f is concave) iff $a \leq 0, b \leq 0$, and $ab \geq c^2$.

Problem 2 (Properties of concave functions I) (Sundaram pg 198) Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be a concave function satisfying $f(0) = 0$. Show that for all $k \geq 1$ we have $kf(x) \geq f(kx)$. What happens if $k \in [0, 1)$? The first part is immediate from Theorem 7.5 on page 179 of Sundaram. If $k < 1$, it is immediate from the definition of concavity that the inequality is reversed.

Problem 3 (Identifying concave and convex functions) (Sundaram, page 198)

Show that the affine function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) = a \cdot x + b$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ is both convex and concave on $\mathbb{R}^n$. Conversely, show that if $f : \mathbb{R}^n \to \mathbb{R}$ is both concave and convex, then it is an affine function.

Problem 4 (Properties of concave functions II) (Sundaram pg 198) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be concave functions.

a. Give an example to show that their composition $f \circ g$ is not necessarily concave.

Consider $g(x) = \sqrt{x}$ and $f(x) = -x$. Clearly, both are concave, yet $f(g(x)) = -\sqrt{x}$, which has positive second derivative and is thus strictly convex (and thus not concave).

b. Is the product $f * g$ concave? Prove or provide a counter example.No. Consider $f(x) = x$, $g(x) = x$. Then $f(x) * g(g) = x^2$, which is strictly convex, and thus not concave.

Problem 5 (Constrained maximization) (Sundaram, pg 200) Let T be any positive integer. Consider the following problem:

$$\begin{aligned} \max \quad & \sum_{t=1}^{T} u(c_t) \\ \text{subject to} \quad & c_1 + x_1 \leq x \\ & c_t + x_t \leq f(x_{t-1}), t = 1, 1, ..., T \\ & c_t, x_t \geq 0, t = 1, 2, ..., T \end{aligned}$$