Midterm

This exam spans the front and back of this sheet. For problems 1-5, points will be awarded primarily based on your explanation of why your answer is correct. Carefully state the appropriate definitions and theorems, where applicable, and argue how they apply. Also, make sure that every step in your argument follows logically and directly from the immediately preceding one. For problems 6-7, your score will be based primarily on your final answers, so do not spend much time on explaining yourself. Good luck.

Problem 1 (15 points) Give an example of an open set in $\mathbb{R}^2$ that is not convex. Make sure to explain in detail why your set is open and why it is not convex.

Problem 2 (20 points) For each of the following, state whether the described function attains a maximum and minimum on the set A. If so, prove it, if not, provide a detailed counterexample.

a. $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^7 - 5x^5$, $A = [-12, 14]$.

b. $f : \mathbb{R} \to \mathbb{R}$ is continuous, with $f(0) = 7$ and $\lim_{x\to\infty} = 0$; $A = [0, \infty)$.

c. $f : \mathbb{R}^2 \to \mathbb{R}$, with $f(x, y) = x^2 + 2xy + y^2 + 6$; $A = \mathbb{R}^2$.

Problem 3 (10 points) True/false: a function $f : \mathbb{R} \to \mathbb{R}$ which does not attain a maximum on $A = (1, 10)$ necessarily does not attain a maximum on $A' = [2, 9]$. Prove or provide a detailed counterexample.

Problem 4 (15 points) A firm produces John McLaughlin action figures using labor and capital. Specifically, if it has L labor and K capital, it can produce $y = f(K, L)$ action figures. Labor costs the firm $w > 0$ per unit, and capital costs $r > 0$. $f(0, 0) = 0$, f is strictly increasing in both variables, continuous, and unbounded.

The firm is interested in determining, for any output level y, the cheapest possible cost of producing y. It is thus interested in minimizing costs $rK + wL$ over the set $\{(K, L) \in \mathbb{R}^2 : f(K, L) \geq y\}$, for any given y. (Hint: that f is continuous implies that $\{(K, L) \in \mathbb{R}^2 : f(K, L) \geq y\}$ is a closed set. You may take this fact as given.)

True/false: the firm's minimization problem necessarily has a solution. Prove or counter.

Problem 5 (10 points) Determine whether the set $B = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = 5\}$ is open, closed, both, or neither in $\mathbb{R}^2$ with the standard metric. Prove your answers.

For problems 6 and 7 only, points will be awarded based primarily on your final numerical answer. Do not prove anything or provide justification for your answers, just write them down.

Problem 6 (20 points) Calculate the following Taylor approximations. Show your work, but do not prove anything.

a. What is the third-order Taylor approximation of $\sin(x^2)$ about $x = 0$?

b. What is the n^{th}-order Taylor approximation of $\cos(x)$ about $x = 0$? Take the limit as $n \to \infty$ to obtain a series representation of the cos function.

c. What is the first-order Taylor approximation of $f(x, y) = 3x^2y + 5y$ around $(x, y) = (1, 1)$?

Problem 7 (10 points) Find the $\limsup$ and $\liminf$ of the following sequences. Give a numerical answer only.

a. $(-1)^{n-1}(1 + \frac{1}{n})$

b. $(-1)^n n$