Midterm

10/16/08

This exam spans the front and back of this sheet. For problems 1-5, points will be awarded primarily based on your explanation of why your answer is correct. Be sure to cite any relevant definitions or results from lecture or a book, and explain your answers in detail. For problems 6-7, your score will be based primarily on your final answers, so do not spend much time on explaining yourself. Good luck.

Problem 1 (15 points) Give an example of an open set in $\mathbb{R}^2$ that is not convex. Make sure to explain in detail why your set is open and why it is not convex.

Consider $O = (0,2)^2 \bigcup (4,6)^2$. This is open (unions of opens sets are open, and for any $(x,y) \in (0,2)^2$, setting ϵ to $\frac{1}{2}\min\{2-x, 2-y\}$ ensures $B((x,y),\epsilon) \subset (0,2)^2$, similar for $(4,6)^2$). It is not convex as $\frac{1}{2}(1,1) + \frac{1}{2}(5,5) = (3,3)$, and $(3,3) \notin O$.

Problem 2 (20 points) For each of the following, state whether the described function attains a maximum and minimum on the set A. If so, prove it, if not, provide a detailed counterexample.

a. $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^7 - 5x^5$, $A = [-12, 14]$. Yes, as f is continuous and A is compact, f attains both a max and a min on A by the Weierstrass theorem.

b. $f : \mathbb{R} \to \mathbb{R}$ is continuous, with $f(0) = 7$ and $\lim_{x\to\infty} = 0$; $A = [0,\infty)$.f attains a max on A; that $\lim_{x\to\infty} f(x) = 0$ implies $f(x) < 1$ for $x > \tilde{x}$ for some $\tilde{x} \in \mathbb{R}$ (this follows from the definition of a limit). Given that $f(0) > 1$ and that f is continuous, the max of f on A is equal to the max of f on $[0,\tilde{x}]$, a closed and bounded set, and thus f indeed attains a max on A. However, f need not have a min; consider $f(x) = \frac{7}{x+1}$; this satisfies the given conditions, but clearly has no min on A.

c. $f : \mathbb{R}^2 \to \mathbb{R}$, with $f(x,y) = x^2 + 2xy + y^2 + 6$; $A = \mathbb{R}^2$. This has no max; to see this, set y to 0 and let x grow; as $x \to \infty$, $f(x,0) \to \infty$. It does, however, have a min. $f(x,y) = (x+y)^2 + 6$, which is always greater than or equal to 6. That $f(0,0) = 6$ implies that $(0,0)$ is a minimizer of f.

Problem 3 (10 points) True/false: a function $f : \mathbb{R} \to \mathbb{R}$ which does not attain a maximum on $A = (1,10)$ necessarily does not attain a maximum on $A' = [2,9]$. Prove or provide a detailed counterexample.False. Consider $f(x) = x$. The clearly attains a max on $[2,9]$ (it is continuous, and $[2,9]$ is compact), yet does not attain a max on $(1,10)$ (to see this, suppose f did attain a max at $x^* \in (1,10)$ and consider $x^{**} = \frac{x^*+10}{2} \in (1,10)$; clearly, $f(x^{**}) > f(x^*)$.

Problem 4 (15 points) A firm produces John McLaughlin action figures using labor and capital. Specifically, if it has L labor and K capital, it can produce $y = f(K,L)$ action figures. Labor costs the firm $w > 0$ per unit, and capital costs $r > 0$. $f(0,0) = 0$, f is strictly increasing in both variables, continuous, and unbounded.

The firm is interested in determining, for any output level y, the cheapest possible cost of producing y. It is thus interested in minimizing costs $rK + wL$ over the set $\{(K,L) \in \mathbb{R}^2 : f(K,L) \geq y\}$, for any given y. (Hint: that f is continuous implies that $\{(K,L) \in \mathbb{R}^2 : f(K,L) \geq y\}$ is a closed set. You may take this fact as given.)

True/false: the firm's minimization problem necessarily has a solution. Prove or counter.

See Sundaram Example 3.8, page 94.

Problem 5 (10 points) Determine whether the set $B = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = 5\}$ is open, closed, both, or neither in $\mathbb{R}^2$ with the standard metric. Prove your answers.

It is not open, as $B((5, 0), \epsilon)$ contains the point $(5+\frac{1}{2}\epsilon, 0) \notin B$. To show it is closed, we show its complement is open. Pick $(x, y) \notin B$. Then, $\sqrt{x^2 + y^2} \neq 5$. Suppose $\sqrt{x^2 + y^2} > 5$, and set $\epsilon = \frac{1}{2}(5 + \sqrt{x^2 + y^2})$. Clearly, then, every point in $B((w, y), \epsilon)$ is more than 5 away from the origin, so that $B((x, y), \epsilon) \subset B^c$. A similar proof works for (x, y) such that $\sqrt{x^2 + y^2} < 5$.

Problem 6 (20 points) Calculate the following Taylor approximations.

a. What is the third-order Taylor approximation of $\sin(x^2)$ about $x = 0$?

$f(x) = x^2$.

b. What is the n^{th}-order Taylor approximation of $\cos(x)$ about $x = 0$? Take the limit as $n \to \infty$ to obtain a series representation of the cos function.

$\cos(x) = \sum_{\tau=0}^{\infty} \frac{(-1)^\tau x^{2\tau}}{(2\tau)!}$

c. What is the first-order Taylor approximation of $f(x, y) = 3x^2y + 5y$ around $(x, y) = (1, 1)$?

$f(x, y) = 6x + 8y - 6$

Problem 7 (10 points) Find the lim sup and lim inf of the following sequences. Give a numerical answer only.

a. $(-1)^{n-1}(1 + \frac{1}{n})$ The $\limsup$ is 1, the $\liminf$ is -1.

b. $(-1)^n n$ The $\limsup$ is ∞, the $\liminf$ is $-\infty$.