# Homework 2

answers

**Please point out any mistakes you find.**

**Problem 1** MWG question 8.B.3 Let $B = \max_{j \neq i} b_j$ be the maximum bid apart from player $i$. If $B > v_i$, clearly bidding below $v_i$ is no better than bidding $v_i$, as either bid will lose the auction. Bidding above $v_i$ might change the outcome of the auction, but this can only make $i$ worse off, as should he win, he will have to pay $B > v_i$, which is worse than losing the auction. Now suppose $B < v_i$. Here, bidding above $v_i$ is clearly no better than bidding $v_i$, and bidding below $v_i$ is either no better (in the case of $b_i \in (B, v_i)$) or strictly worse (in the case of $b_i < B$. As other players' bids were left arbitrary, we can conclude that no bid other than $v_i$ could ever increase $i's$ payoff, regardless of the behavior of other bidders.

**Problem 2** A newspaper runs the following contest: Each participant mails in a postcard on which he writes an integer between 0 and 1000 (inclusive). Given the entries, the *target integer* is defined to be $\frac{9}{10}$ times the highest entry, rounding downward if the result is not an integer. All participants who chose the target integer split a $10,000 prize.

Suppose this contest is modeled as a simultaneous move game among 100 players. Using a solution concept developed in this course, determine a unique prediction of play. State the weakest possible assumptions about the players' knowledge and abilities which would justify your prediction. A number above 900 is not a best response to any combination of other players' strategies. But then rational players who believe all other players are also rational will not play any number above 810, as they will expect that no opponent will play a number above 900. But then rational players who believe their opponents rational, and who believe that their opponents believe all other opponents are rational will not play any number above 739, and so on. If the game has common knowledge of rationality, the only strategy profile surviving iterated removal of strategies which are never a best response is all players playing the number 0. Note this is also the game's unique Nash equilibrium.

**Problem 3** MWG question 8.D.4 For simplicity, restrict each player's strategy set to the interval $[0, 100]$ (otherwise we have nonsensical results like each player offering $1M being a Nash equilibrium). Note that any strategy outside of this interval would be weakly dominated.

a. No. No player can ever do worse than receiving $0, and any strategy induces this outcome against some subset of the opponent's strategy set.

b. 0 is a weakly dominated strategy; it always gives the minimum payoff of 0, whereas any other strategy in $[0, 100]$ gives a higher payoff against at least a subset of the opponent's strategy set. No other strategy is weakly dominated. Why? Consider $x \in (0, 100]$. Against the opponent's strategy $100 - x$, $x$ gives a strictly higher payoff than any other strategy.

c. The profile $(x, 100 - x)$ is a Nash equilibrium for every $x$.

**Problem 4** MWG question 8.C.4

Given the correction from my email, we should start by subtracting $\pi$ from all payoffs and dividing all payoffs by $\eta$. Letting $x = \frac{4\epsilon}{\eta} > 1$, the payoffs can be rewritten as:

| | | | 3 | |
|---|---|---|---|---|
| | | | $l$ | $r$ |
| 2 | $U$ | | $x, -1, -x$ | $-x, \frac{1}{2}, x$ |
| | $D$ | | $x, \frac{1}{2}, -x$ | $-x, -1, x$ |

The easiest way to see part a is to draw a picture. See the figure on the final page of this file, which plots player 3's utility as a function of the actions of 1 and 2. If 2 plays $U$, $M$ is never a best response for 1. Same if 2 plays $D$. A mathematical version of this picture would also be entirely appropriate.

To show that $M$ is not strictly dominated for part b., note that from the figure, $M$ is clearly not dominated by either $L$ or $R$. Were it to be dominated by a mixture of $L$ and $R$, we would need $u_3(\alpha L+(1-\alpha)R, \sigma_{-3}) > u_3(M, \sigma_{-3})$ for all $\sigma_{-3}$. Consider first the profile in which 2 plays $D$ and 3 plays $l$. $M$ being inferior here would require

$$\alpha x + (1-\alpha)(-x) > \frac{1}{2}$$

$$\Rightarrow \alpha > \frac{1}{2} + \frac{1}{4x} \tag{1}$$

while $M$ being worse than the same mixture against 2 playing $U$ and 3 $r$ would require

$$\alpha(-x) + (1-\alpha)x > \frac{1}{2}$$

$$\Rightarrow \alpha < \frac{1}{2} - \frac{1}{4x} \tag{2}$$

As (1) and (2) are mutually contradictory, we conclude that $M$ is not a dominated strategy.

Finall, for c, note that if players 2 and 3 play $(D, l)$ 50% of the time and $(U, r)$ 50% of the time, $M$ is a best response for player 3.

**Problem 5** Compute all Nash equilibria of the reduced normal form of the game in Figure 1 of HW1.

There do not appear to be any strictly dominated strategies to eliminate. We therefore need to consider each of the 27 supports separately. We can do this most easily by going one by one through each of the 9 supports for 2 of the players, say 2 and 3:

$(a, L)$: 1 prefers $A$, in which case neither 2 nor 3 prefers to switch. Equilibrium.

$(d, L)$: 1 is indifferent between $A$ and $D$. If 1 puts positive weight on $A$, 2 wants to switch to a. If 1 plays only $D$, neither 2 nor 3 prefer to switch. Equilibrium.

$(ad, L)$: 1 prefers $A$, in which case 2 prefers to put all weight on $a$. No equilibrium.

$(a, R)$: 1 prefers $A$, in which case 2 prefers to switch to $d$. No equilibrium.

$(d, R)$: 1 prefers $D$, in which case 3 prefers to switch to $L$. No equilibrium.

$(ad, R)$: If 3 plays $R$, 2 prefers to put all weight on $d$ regardless of what 1 does. No equilibrium.

$(a, LR)$: 1 prefers $A$. 3 is then indifferent between $L$ and $R$. 2 prefers $a$ so long as $\sigma_3(L) \geq \frac{1}{3}$. Equilibrium.

$(d, LR)$: 1 prefers $D$. Then 3 prefers to put all weight on $L$. No equilibrium.

$(ad, LR)$: Suppose 1 plays only $A$. Then 3 prefers to put all weight on $R$, and there is no equilibrium. Now suppose 1 plays $D$. Then 3 prefers to put all weight on $L$, and there is no equilibrium. Suppose 1 mixes. Let $L = P(3 \text{ plays } L)$, $a = P(2 \text{ plays } a)$, and $A = P(1 \text{ plays } A)$. Then, 1 is indifferent over $A$ and $D$ if $2a = 1 - L$. 2 is indifferent between $a$ and $d$ if $L = \frac{1}{3}$. 1's indifference condition then requires $a = \frac{1}{3}$. Finally, 3 is indifferent over $L$ and $R$ if $A = \frac{3}{7}$. Conculde there is an equilibrium if 1 plays $\frac{3}{7}A + \frac{4}{7}D$, 2 plays $\frac{1}{3}a + \frac{2}{3}d$, and 3 plays $\frac{1}{3}L + \frac{2}{3}R$.

Therefore, the Nash equilibria of this game are located at $(A, a, L)$, $(D, d, L)$, $(A, a, \sigma_3(L) \geq \frac{1}{3})$, and $(\frac{3}{7}A + \frac{4}{7}D, \frac{1}{3}a + \frac{2}{3}d, \frac{1}{3}L + \frac{2}{3}R)$.

**Problem 6** In normal form game $G = \{I, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}\}$, can a strategy which places positive probability on more than one pure strategy be strictly dominant? Provide an example, or prove as rigorously as you are able that this cannot occur.

A strategy which is not pure cannot be strictly dominant. To see this, suppose that the strictly dominant mixed strategy $\sigma_i$ places probability $\alpha$ on strategy $s_i^1$ and $1-\alpha$ on $s_i^2$. By virtue of $\sigma_i$ being strictly dominant, the following must hold:

$$u_i(\sigma_i, \sigma_{-i}) > u_i(s_i^1, \sigma_{-i}) \text{ for all } \sigma_{-i} \tag{3}$$

$$u_i(\sigma_i, \sigma_{-i}) > u_i(s_i^2, \sigma_{-i}) \text{ for all } \sigma_{-i} \tag{4}$$

But this implies $u_i(\sigma_i, \sigma_{-i}) > \alpha u_i(s_i^1, \sigma_{-i}) + (1-\alpha)u_i(s_i^2, \sigma_{-i})$. However, by definition of a mixed strategy, $u_i(\sigma_i, \sigma_{-i}) = \alpha u_i(s_i^1, \sigma_{-i}) + (1-\alpha)u_i(s_i^2, \sigma_{-i})$, a contradiction. Conclude that a strictly dominant mixed strategy is impossible. This argument generalizes easily to mixed strategies that place weight on more than two strategies.

**Problem 7** Consider the following game:

| | | 2 | |
|---|---|---|---|
| | | $g$ | $b$ |
| 1 | $G$ | $3,3$ | $0,5$ |
| | $B$ | $5,0$ | $-4,-4$ |

a. Draw a picture of the best response correspondences $b_1 : \Sigma_2 \rightrightarrows \Sigma_1$ and $b_2 : \Sigma_1 \rightrightarrows \Sigma_2$. (Hint: when $\Sigma_2$ contains three pure strategies, the domain of $b_1$ can be represented by an equilateral triangle. What is the anaolgue for two pure strategies?)

b. Find all Nash equilibria of this game.

**Problem 8** Compute all Nash equilibria of the symmetric normal form game below. (Hint: begin by drawing the best response correspondence $b_1 : \Sigma_2 \rightrightarrows \Sigma_1$. Do not skip this step.)

| | | | 2 | |
|---|---|---|---|---|
| | | $L$ | $C$ | $R$ |
| | $T$ | $0,0$ | $6,-3$ | $-4,-1$ |
| 1 | $M$ | $-3,6$ | $0,0$ | $5,3$ |
| | $B$ | $-1,-4$ | $3,5$ | $0,0$ |

See figures for pictures of the best response correspondences. To compute Nash equilibria, we'll look at each possible supports for player 1:

$T$: 2 prefers $L$. Given this, 1 prefers $T$. Equilibrium.

$M$: 2 prefers $L$. Given this, 1 prefers $T$. No equilibrium.

$B$: 2 prefers $C$. Given this, 1 prefers $T$. No equilibrium.

$TM$: 2 prefers $L$. Given this, 1 prefers $T$ only. No equilibrium.

$TB$: 2 could play $L$, $R$, $C$, $LR$ or $RC$, depending on the mixture. Look at each of these severally:

- $L$: 1 prefers $T$ only. No equilibrium.
- $R$: 1 prefers $M$. No equilibrium.
- $C$: 1 prefers $T$ only. No equilibrium.
- $LR$: requires 1 playing $\frac{4}{5}T + \frac{1}{5}B$. 1 is willing to do this if 2 plays $\frac{4}{5}L + \frac{1}{5}R$. Equilibrium.
- $RC$: Given this support, 1 prefers either $T$, $M$, or a mixture of $T$ and $M$. No equilibrium.

$MB$: For 1 to be willing to mix between $M$ and $B$, 2 must assign prositive probability to $R$, but 2 will not do this if 1 mixes between $M$ and $B$.

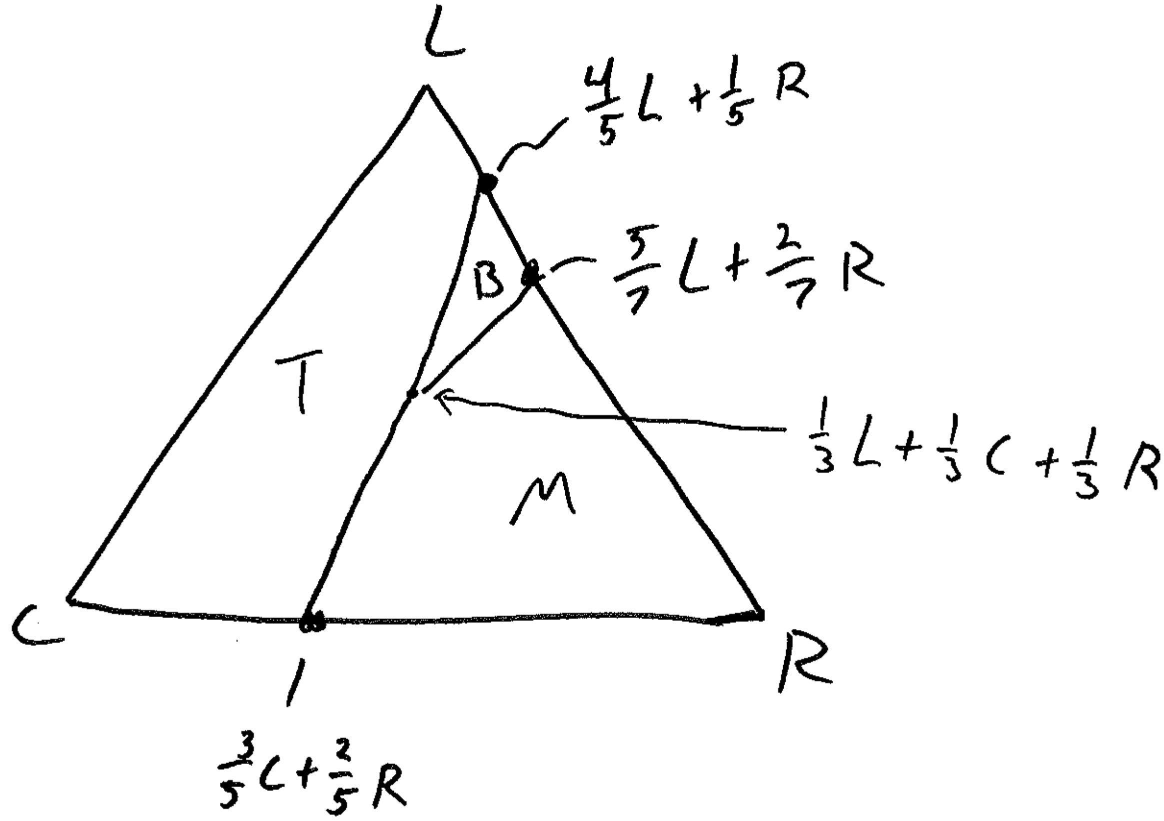
$TMB$: equilibrium with both putting $\frac{1}{3}$ probability on each of their strategies.

So there are 3 Nash equilibria: $(T, L)$, $(\frac{4}{5}R + \frac{1}{5}B, \frac{4}{5}L + \frac{1}{5}R)$, and $(\frac{1}{3}T + \frac{1}{3}M + \frac{1}{3}B, \frac{1}{3}L + \frac{1}{3}C + \frac{1}{3}R)$.

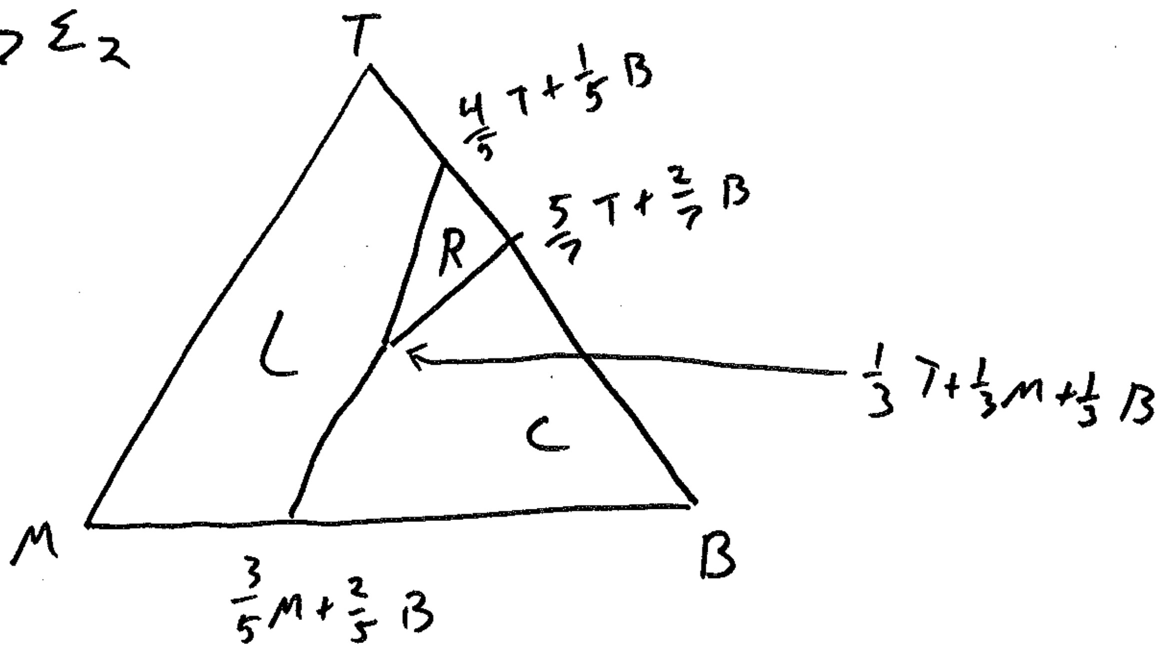
**Optional question for personal enrichment** A town has 100 voters: 51 conservatives and 49 liberals. A conservative and a liberal candidate are running for mayor. Voting is by simple majority, and in the case of a tie assume the liberal candidate wins. A conservative voter gets a payoff of 10 if the conservative candidate is elected and -10 if the liberal is elected; vice versa for a liberal voter. It costs a citizen 1 to vote.

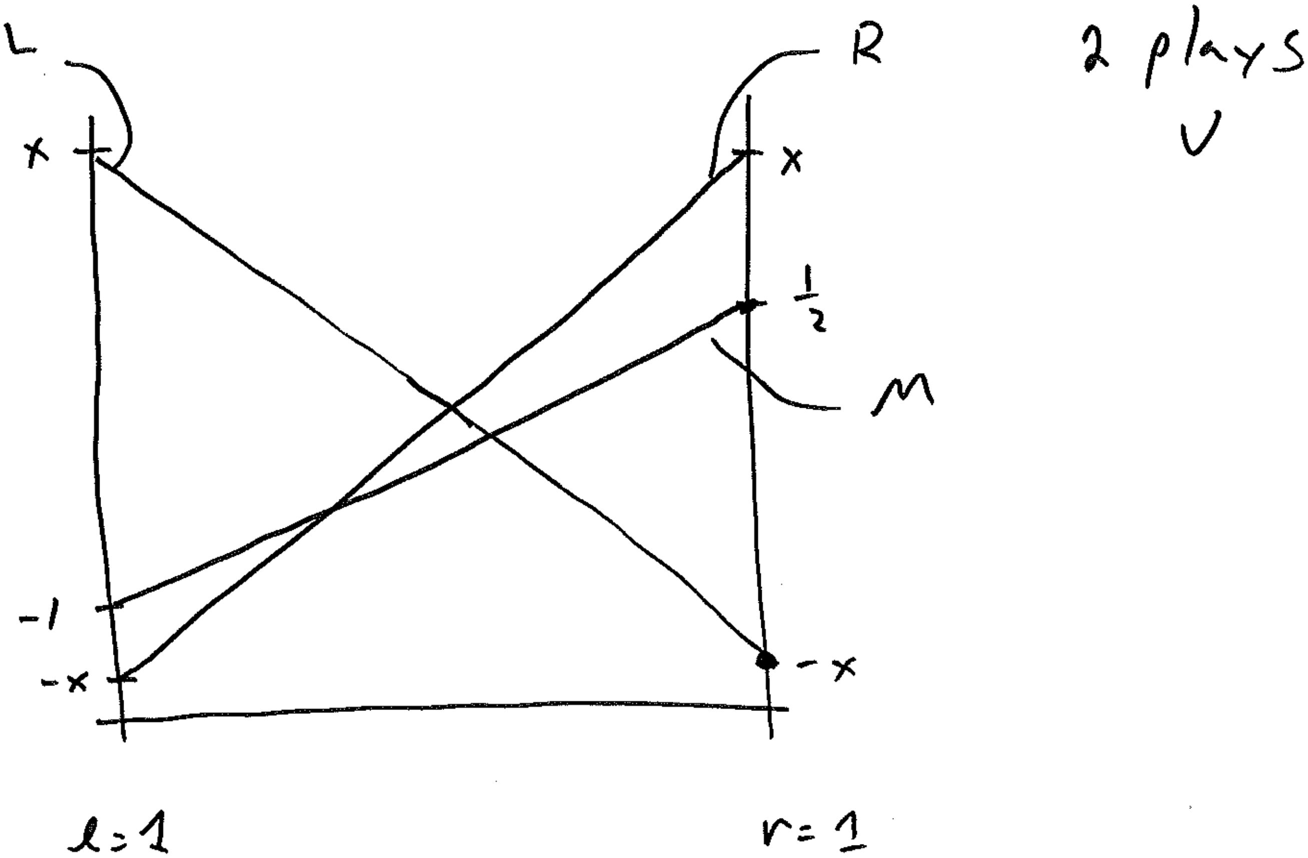
Solve for the Nash equilibrium of this game. If you cannot explicitly solve the model, equations characterizing the equilibrium are good as well. Let me know if you get a good answer.

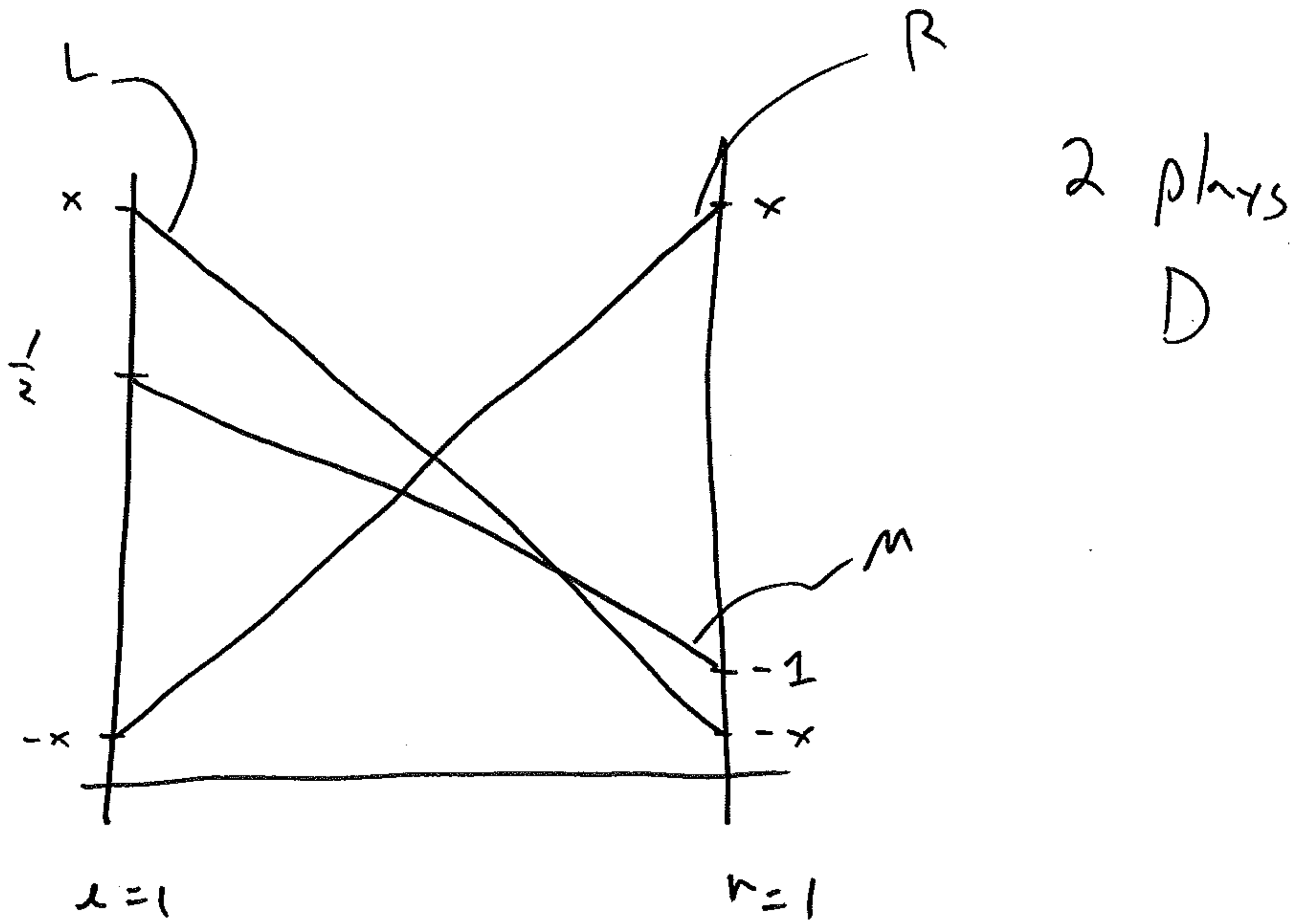
$BR_1 : \Sigma_2 \Rightarrow \Sigma_1$




$BR_2 : \Sigma_1 \Rightarrow \Sigma_2$






1's utility as function of 2,3's strategy.