# Homework 4

answers

**Problem 1** In the signaling games in figures 7 and 8, compute all sequential equilibria and determine which of them satisfy the intuitive criterion.

Starting with figure 7, note that $B$ is dominated (by $\frac{1}{2}T + \frac{1}{2}M$ for player 2, and so won't be played in any equilibrium. Given this, $m_2$ is dominated for $t_2$, and so $t_2$ plays $m_1$ in every equilibrium. There are two sets of sequential equilibria. In the first, $\sigma_1(m_2|t_1) = 1 = \sigma_1(m_1|t_2)$, and $\sigma_2(T|m_2) = 1$. In the second, $\sigma_1(m_1|t_1) = \sigma_1(m_1|t_2) = 1$, and either $\sigma_2(M|m_2) = 1$ with $\mu(t_2|m_2) \geq \frac{1}{2}$, or $\sigma_2(M|m_2) \geq \frac{2}{3}$ (and $\sigma_2(B|m_2) = 0$) with $\mu(t_2|m_2) = \frac{1}{2}$.

Since the first equilibrium does not involve an unused message, it trivially satisfies the Intuitive Criterion. The intuitive criterion does eliminate the second set of equilibria. Once player 2 is required to play a best response (i.e. $T$ or $M$), $t_2$'s equilibrium payoff of 0 dominates the payoffs he might obtain by deviating to $m_2$. Therefore, player 2 must not believe that she is facing $t_2$, and must therefore play $T$. In this case, $t_1$ benefits by deviating to $m_2$, breaking the equilibrium. Formally,

$$
\begin{aligned}
&T(m_2) = \{t_1, t_2\} \\
&D(m_2) = \left\{ t \in T(m_2) : u_1^*(t) > \max_{r \in BR(T(m_2), m_2)} u_1(t, m_2, r) \right\} = \{t_2\} \\
&BR(T(m_2) \backslash D(m_2), m_2) = BR(\{t_1\}, m_2) = \{T\} \\
&u_1^*(t_1) = 0 < 2 = u_1(t_1, m_2, T) \qquad (1)
\end{aligned}
$$

(1) implies that the intuitive criterion is violated.

For figure 8, once again, there are two sets of sequential equilibria. In any equilibrium, 2 does not play the dominated strategy $D$. In the first set of equilibria, type $t_1$ mixes between $m_1$ and $m_2$, type $t_2$ plays $m_2$, and type $t_3$ plays message $m_1$, and 2 mixes between $T$ and $M$.

In the other equilibria, all types choose $m_1$. This can be supported by a number of combinations of behavior and beliefs for player 2:

$$
\begin{aligned}
&\sigma_2(B|m_2) = 1, \text{ with } \mu(t_3|m_2) \geq \mu(t_1|m_2) \text{ and } \mu(t_3|m_2) \geq \mu(t_2|m_2); \\
&\sigma_2(B|m_2) \geq \frac{1}{2} \text{ and } \sigma_2(M|m_2) = 0, \text{ with } \mu(t_3|m_2) = \mu(t_1|m_2) \geq \mu(t_2|m_2); \\
&\sigma_2(B|m_2) \geq \frac{1}{2} \text{ and } \sigma_2(T|m_2) = 0, \text{ with } \mu(t_3|m_2) = \mu(t_2|m_2) \geq \mu(t_1|m_2); \\
&\sigma_2(B|m_2) \geq \sigma_2(T|m_2) \text{ and } \sigma_2(B|m_2) \geq \sigma_2(M|m_2), \text{ with } \mu(t_3|m_2) = \mu(t_2|m_2) = \mu(t_1|m_2) = \frac{1}{3}
\end{aligned}
$$

(To see what is going on, first draw the simplex representing 2's possible mixed responses to $m_2$, and graph the regions where $m_1$ is a best response for both $t_1$ and $t_2$. Then, draw another simplex representing 2's possible beliefs after receiving $m_2$, and graph the regions where $T$, $M$, and $B$ are her best responses. The four classes of equilibria above come from matching appropriate regions in the two diagrams.)

Since the first set of equilibria have no unused messages, we need only check the second set against the Intuitive Criterion. Note that for $t_3$, all payoffs to $m_2$ are dominated by the equilibrium payoff of 0; hence, $D(m_2) = \{t_3\}$. Player 2's best responses to the types that remain are $T$ and $M$. If 2 plays $T$, then $t_1$ does not strictly improve his payoff by deviating, while if 2 plays $M$, $t_2$ does not strictly improve his payoff by deviating. Thus, the equilibria satisfy the Intuitive Criterion, since there is no single type who is better of deviating regardless of the best response played by player 2.

**Problem 2** In the signaling game in figure 9, find a sequential equilibrium in which message 2 is not played. Does this equilibrium satisfy the iterated intuitive criterion?

One sequential equilibrium in which all players choose message $m_1$ has player 2 play $T$ against $m_2$, and has her believe $\mu(t_1|m_2) = 1$. There are many other sequential equilibria in which message $m_2$ is unused, but all of them are eliminated by the argument that follows. Since $m_2$ is dominated for type 1, 2 should believe $\mu(t_1|m_2) = 0$. Given this, it is no longer a best response for player 2 to play $T$. Therefore, $t_2$ should not want to play $m_2$ and so 2 should now believe $\mu(t_2|m_2) = 0$, meaning that it must be that $\mu(t_3|m_2) = 1$, and so 2 must play $B$, which means $t_3$ would be better off by deviating to $m_2$. Thus, all equilibria in which all players choose message $m_2$ violate the Intuitive Criterion. Formally,

$$T(m_2) = \{t_1, t_2, t_3\}$$
$$D(m_2) = \left\{ t(m_2) : u_1^*(t) > \max_{r \in BR(T(m_2), m_2)} u_1(t, m_2, r) \right\} = \{t_1\}$$
$$BR(T(m_2)\backslash D(m_2), m_2) = BR(\{t_2, t_3\}, m_2) = \{M, B\}$$

Iterating,

$$D'(m_2) = \left\{ t \in T(m_2)\backslash D(m_2) : u_1^*(t) > \max_{r(T(m_2)\backslash D(m_2), m_2)} u_1(t, m_2, r) \right\} = \{t_2\}$$
$$BR(T(m_2)\backslash D(m_2)\backslash D'(m_2), m_2) = \{B\}$$
$$u_1^*(t_3) = 0 < 1 = u_1(t_3, m_2, B)$$

**Problem 3** Consider infinite repetitions of the games in figures 1, 2, and 3. In each case, sketch the set of payoff vectors attainable in a subgame perfect equilibrium, and those which are attainable in a subgame perfect equilibrium using Nash reversion strategies. (Suppose that the discount rate is very close to one.)

Figure 1: See sketch below. All payoffs in $F \bigcap IR$ are supportable in a subgame perfect equilibrium if $\delta$ is high enough, while the stage game's sole Nash equilibrium has payoffs $(2, 2)$. As there are no feasible payoff vectors that dominate $(2, 2)$, Nash reversion is useless in this game. Note that 1 and 2 both have minmax payoffs of 1.

Figure 2: 1's minmax payoff is 1, 2's is 0. The stage game's lone Nash equilibrium is $(\frac{1}{2}T + \frac{1}{2}B, \frac{1}{2}l + \frac{1}{2}r)$, which gives payoffs $(2, 2)$. See sketch below for a picture of payoffs supportable using Nash reversion and supportable in some SPE using carrot and stick strategies, if $\delta$ is high enough..

Figure 3: 1's minmax payoff is $\frac{7}{2}$, while 2's is $\frac{1}{2}$. There is one Nash equilibrium, at $(T, R)$, with payoffs $(4, 1)$. See sketch below for strategies supportable in a SPE using Nash reversion and carrot and stick strategies, if $\delta$ is high enough..

**Problem 4** Consider an infinite repetition of the normal form game in figure 4. For what values of $\delta$ can the play path $\{(C, C), (C, C), ...\}$ be supported in a Nash equilibrium? What about in a subgame perfect equilibrium?

Since the game is symmetric, we need check incentive compatibility for only one player. Under grim trigger strategies with mutual minmaxing as the punishment path, player 1 prefers not to deviate if

$$\begin{aligned} 2 &\geq (1-\delta)8 + \delta 1 \\ \iff \delta &\geq \frac{6}{7} \end{aligned}$$

Since mutual minmaxing coincides with a Nash equilibrium (a unique feature of the prisoner's dilemma), in this case no player will want to deviate from a punishment path of perpetual play of the Nash equilibrium $(D, D)$, and so the grim trigger strategy is also a SPE so long as $\delta \geq \frac{6}{7}$.

**Problem 5** Consider an infinite repetition of the normal form game in figure 5.

**a.** Show that payoffs of $(4, 4)$ can be supported in a subgame perfect equilibrium using a Nash reversion strategy if and only if $\delta \geq \frac{1}{2}$.

The unique stage game equilibrium is at $(\frac{3}{4}A + \frac{1}{4}B, \frac{1}{2}a + \frac{1}{2}b)$, with payoffs of $(3, \frac{7}{4})$. Suppose the players play $(B, b)$ on the equilibrium path, with a punishment path of Nash reversion. As $b$ is 2's best response to $B$, clearly only 1 would consider deviating from the equilibrium path. 1 prefers not to deviate iff

$$\begin{aligned} 4 &\geq 5(1-\delta) + 3\delta \\ \iff \delta &\geq \frac{1}{2} \end{aligned}$$

since the punishment path involves Nash reversion, no player wants to deviate once the punishment path is reached, and so Nash reversion strategies comprise a SPE here if $\delta \geq \frac{1}{2}$.

**b.** Show that for every $\delta \geq \frac{1}{4}$, there is a subgame perfect strategy profile yielding payoffs of $(4, 4)$.

Finding workable carrot and stick strategies requires some experimentation. An obvious place to start is to have a punishment path in which both players minmax each other for one period, and then return to the equilibrium path. Therefore, consider the following strategies:

"Play $(B, b)$ initially. If $(B, b)$ was played in period $t-1$, play $(B, b)$ in period $t$. If $(C, c)$ was played in period $t-1$, play $(B, b)$ in period $t$. If anything else was played in $t-1$, play $(C, c)$."

(Note that this strategy amounts to a carrot and stick strategy in which players minmax each other for one period after any deviation from the equilibrium path.) Player 2 will not want to deviate from the equilibrium path. Player 1 will not want to deviate from the equilibrium path if:

$$\begin{aligned} (B, b), (B, b), (B, b), \ldots &\succeq (A, b), (C, c), (B, b), (B, b), \ldots \\ 4 &\geq 5(1-\delta) + 4\delta^2 \\ \iff \delta &\in [\frac{1}{4}, 1] \end{aligned}$$

Finally, we need to check that neither player wants to deviate on the punishment path. Since their incentives are symmetric here, it suffices to check only player 1, who prefers to deviate from the punishment path iff

$$\begin{aligned} (C, c), (B, b), (B, b), \ldots &\succeq (A, c), (C, c), (B, b), (B, b), \ldots \\ 4\delta &\geq 1 - \delta + 4\delta^2 \\ \iff \delta &\in [\frac{1}{4}, 1] \end{aligned}$$

Q.E.D.

**Problem 6** Consider the normal form game $G$ in figure 6 below.

**a.** Determine the set of Nash equilibria of the normal form game.

$G$ is a zero-sum game. Therefore, the easiest way to find its Nash equilibria is to look for the strategies 2 uses to minmax 1. From inspection of player 1's best response correspondence, this is evidently at $\frac{5}{11}X + \frac{5}{11}Y + \frac{1}{11}Z$, giving 1 a minmax payoff of $\frac{51}{11}$. For 2 to be willing to mix between $X$, $Y$, and $Z$, 1 evidently needs to play the mixture $\frac{13}{77}B + +\frac{8}{77}C + \frac{8}{11}E$. Conclude there is a unique Nash equilibrium, at $(\sigma_1, \sigma_2) = (\frac{13}{77}B + +\frac{8}{77}C + \frac{8}{11}E, \frac{5}{11}X + \frac{5}{11}Y + \frac{1}{11}Z)$.

**b.** Let $G^\infty(\frac{3}{4})$ be an infinite repetition of $G$ with common discount rate $\frac{3}{4}$. Sketch both the set of feasible payoffs of $G^\infty(\frac{3}{4})$ and the set of payoffs which are sustainable in some subgame perfect equilibrium of $G^\infty(\frac{3}{4})$.

In this special case of a zero-sum game, the set of feasible payoffs is a straight line, and so $F \bigcap IR$ is a singleton. Therefore, the set of payoffs attainable in an equilibrium of the repeated game is no different from the set of payoffs attainable in the one-shot game.

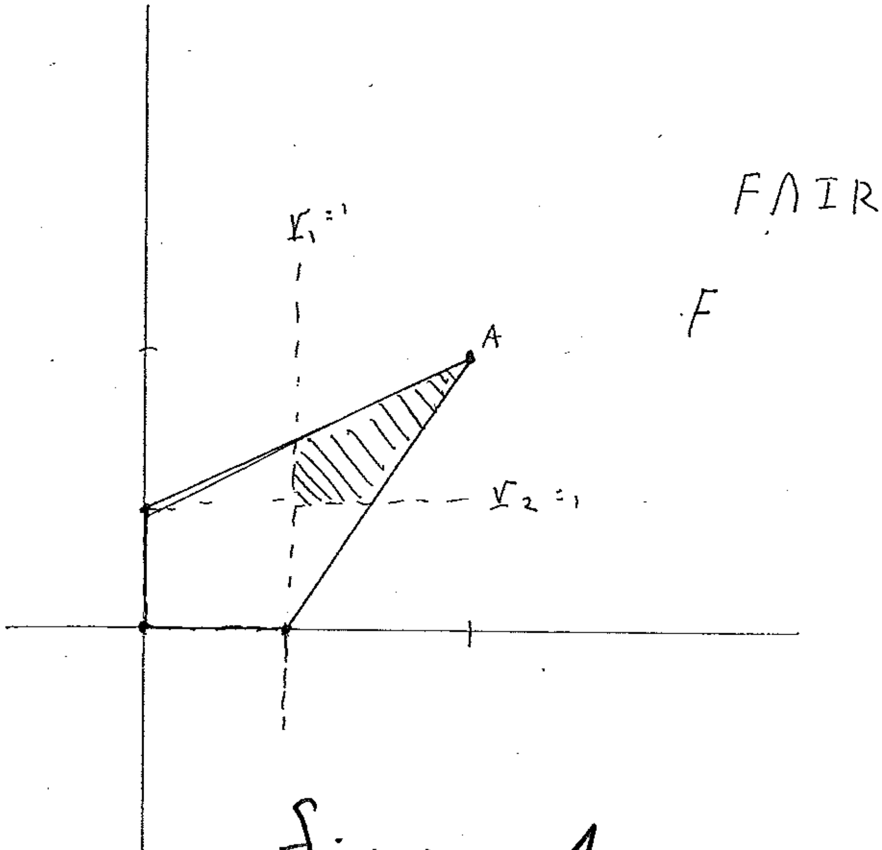


figure 1

Now,

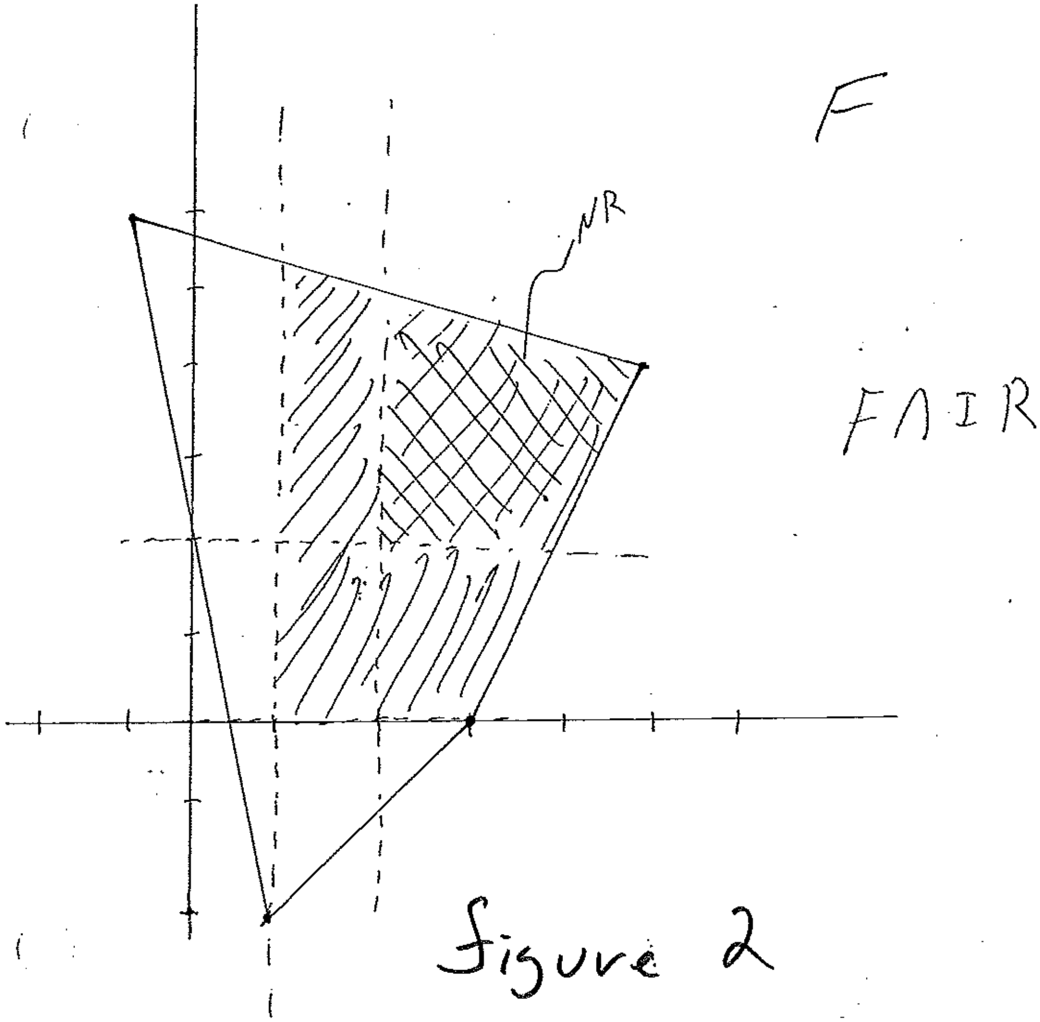


figure 2

Now,

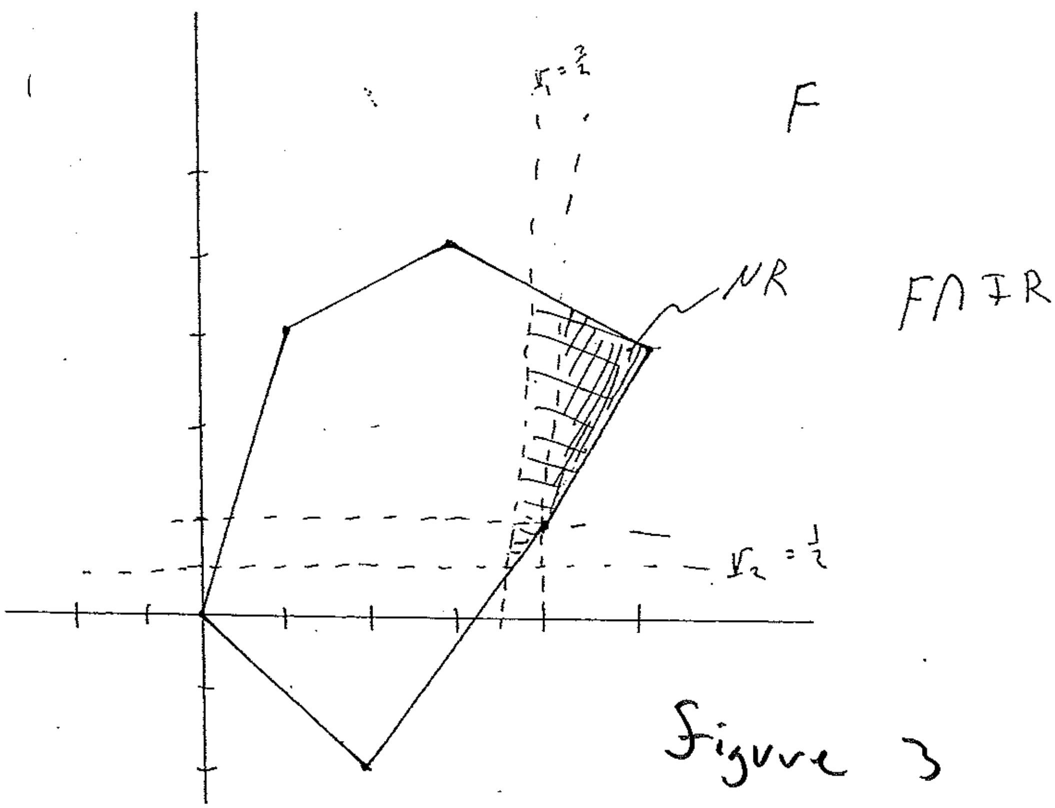


figure 3