Homework 2

due 2/8/2012

Problem 1 A newspaper runs the following contest: Each participant mails in a postcard on which he writes an integer between 0 and 1000 (inclusive). Given the entries, the *target integer* is defined to be $\frac{9}{10}$ times the highest entry, rounding downward if the result is not an integer. All participants who chose the target integer split a $10,000 prize.

a. Suppose this contest is modeled as a simultaneous move game among 100 players. Using only common knowledge of rationality, determine a unique prediction of play.

A number above 900 is not a best response to any combination of other players' strategies. But then rational players who believe all other players are also rational will not play any number above 810, as they will expect that no opponent will play a number above 900. But then rational players who believe their opponents rational, and who believe that their opponents believe all other opponents are rational will not play any number above 739, and so on. If the game has common knowledge of rationality, the only strategy profile surviving iterated removal of strategies which are never a best response is all players playing the number 0. Note this is also the game's unique Nash equilibrium.

b. If you entered such a contest, what number would you personally play, and why?

Problem 2 Consider a 3-player, simultaneous move game with $S_1 = \{L, M, R\}$, $S_2 = \{U, D\}$, and $S_3 = \{l, r\}$. Figure 1 gives **player 1's payoffs** from each of his three pure strategies conditional on the strategy choices of players 2 and 3. So, for example, if 2 plays U and 3 plays l, $u_1(L) = \pi + 4\epsilon$, $u_1(M) = \pi - \eta$, and $u_1(R) = \pi - 4\epsilon$. Assume that *pi*, ϵ, and η are strictly greater than 0, and that $\eta < 4\epsilon$.

		Player 3's strategy	
		l	r
Player 2's strategy	U	$\pi + 4\epsilon, \pi - \eta, \pi - 4\epsilon$	$\pi - 4\epsilon, \pi + \frac{\eta}{2}, \pi + 4\epsilon$
	D	$\pi + 4\epsilon, \pi + \frac{\eta}{2}, \pi - 4\epsilon$	$\pi - 4\epsilon, \pi - \eta, \pi + 4\epsilon$

Figure 1: Player 1's payoffs $(u_1(L), u_1(M), (u_1(R))$ are depicted.

a. Argue that pure strategy M is never a best response for player 1 to any mixed strategy combinations for players 2 and 3.[1]

b. Show that pure strategy M is not strictly dominated for player 1.

c. Which (generically) eliminates more strategies? Iterated removal of strictly dominated strategies or iterated removal of non-rationalizable strategies?

To simplify the problem, start by subtracting π from all payoffs and dividing all payoffs by η. Letting $x = \frac{4\epsilon}{\eta} > 1$, the payoffs can be rewritten as:

[1]Hint: this problem is from MWG, and I'm not sure why three variables are needed to write player 1's utilities. I would start by simplifying player 1's utility. To do so, remember that VNM utility functions are only unique to affine transformations.

		3: l	r
2	U	$x, -1, -x$	$-x, \frac{1}{2}, x$
	D	$x, \frac{1}{2}, -x$	$-x, -1, x$

The easiest way to see part a is to draw a picture. See the figure on the final page of this file, which plots player 3's utility as a function of the actions of 1 and 2. If 2 plays U, M is never a best response for 1. Same if 2 plays D. A mathematical version of this picture would also be entirely appropriate.

To show that M is not strictly dominated for part b., note that from the figure, M is clearly not dominated by either L or R. Were it to be dominated by a mixture of L and R, we would need $u_3(\alpha L+(1-\alpha)R, \sigma_{-3}) > u_3(M, \sigma_{-3})$ for all σ_{-3}. Consider first the profile in which 2 plays D and 3 plays l. M being inferior here would require

$$\alpha x + (1-\alpha)(-x) > \frac{1}{2}$$
$$\Rightarrow \alpha > \frac{1}{2} + \frac{1}{4x} \tag{1}$$

while M being worse than the same mixture against 2 playing U and 3 r would require

$$\alpha(-x) + (1-\alpha)x > \frac{1}{2}$$
$$\Rightarrow \alpha < \frac{1}{2} - \frac{1}{4x} \tag{2}$$

As $\mathring{S}$arah and $\mathring{P}$alin are mutually contradictory, we conclude that M is not a dominated strategy.

Finall, for c, note that if players 2 and 3 play (D, l) 50% of the time and (U, r) 50% of the time, M is a best response for player 3.

Problem 3 Find all Nash equilibria of the normal form game in figure 2:

		2: a	b	c
	A	2, 4	10, 2	2, 0
1	B	4, 2	8, 8	0, 0
	C	0, 2	0, 0	0, 0

Figure 2: game for problem 3

C is strictly dominated by A for player 1, and c is strictly dominated by a for player 2. Removing these strategies, we have a 2x2 game with no pure equilibria and one mixed equilibrium, at $(\frac{3}{4}A + \frac{1}{4}B, \frac{1}{2}a + \frac{1}{2}b)$.

Problem 4 Consider the simultaneous-move game in figure 3.

		2: a	b	c
	x	6, 0	0, 1	0, 1
1	y	0, 1	6, 0	0, 1
	z	5, 1	5, 1	5, 0

Figure 3: game for problem 4

a. Draw the best response correspondences for player 1 and player 2.

See figure 3

b. Describe the set of rationalizable strategies for player 1 and player 2.

For player 1, x, y, z, all mixtures of x and z, and all mixtures of y and z are rationalizable. For player 2, all of Σ_2 is rationalizable.

c. Find all Nash equilibria of this game.

Consider the rationalizable supports for player 1 one at a time:

x: 2 plays b, c, or a mixture. But in response to any of these, 1 plays y, z, or a mixture. Not Nash.

y: 2 plays a, c, or a mixture. But in response to any of these, 1 plays x or z. Not Nash.

z: 2 plays a, b, or a mixture. In response to any of these, 1 plays x, y, z, xz, or yz. Nash equilibrium where 1 plays z, 2 plays $\alpha a + (1-\alpha)b$ for $\alpha \in [\frac{1}{6}, \frac{5}{6}]$.

xz: 2 plays b, but in response to this, 1 plays y. Not Nash.

yz: 2 plays a, but in response to this, 1 plays x. Not Nash.

Problem 5 Compute all Nash equilibria of the reduced normal form of the game in figure 1 of HW1.

There do not appear to be any strictly dominated strategies to eliminate. We therefore need to consider each of the 27 supports separately. We can do this most easily by going one by one through each of the 9 supports for 2 of the players, say 2 and 3:

(a, L): 1 prefers A, in which case neither 2 nor 3 prefers to switch. Equilibrium.

(d, L): 1 is indifferent between A and D. If 1 puts positive weight on A, 2 wants to switch to a. If 1 plays only D, neither 2 nor 3 prefer to switch. Equilibrium.

(ad, L): 1 prefers A, in which case 2 prefers to put all weight on a. No equilibrium.

(a, R): 1 prefers A, in which case 2 prefers to switch to d. No equilibrium.

(d, R): 1 prefers D, in which case 3 prefers to switch to L. No equilibrium.

(ad, R): If 3 plays R, 2 prefers to put all weight on d regardless of what 1 does. No equilibrium.

(a, LR): 1 prefers A. 3 is then indifferent between L and R. 2 prefers a so long as $\sigma_3(L) \geq \frac{1}{3}$. Equilibrium.

(d, LR): 1 prefers D. Then 3 prefers to put all weight on L. No equilibrium.

(ad, LR): Suppose 1 plays only A. Then 3 prefers to put all weight on R, and there is no equilibrium. Now suppose 1 plays D. Then 3 prefers to put all weight on L, and there is no equilibrium. Suppose 1 mixes. Let $L = P(3 \text{ plays } L)$, $a = P(2 \text{ plays } a)$, and $A = P(1 \text{ plays } A)$. Then, 1 is indifferent over A and D if $2a = 1 - L$. 2 is indifferent between a and d if $L = \frac{1}{3}$. 1's indifference condition then requires $a = \frac{1}{3}$.

Finally, 3 is indifferent over L and R if $A = \frac{3}{7}$. Conculde there is an equilibrium if 1 plays $\frac{3}{7}A + \frac{4}{7}D$, 2 plays $\frac{1}{3}a + \frac{2}{3}d$, and 3 plays $\frac{1}{3}L + \frac{2}{3}R$.

Therefore, the Nash equilibria of this game are located at (A, a, L), (D, d, L), $(A, a, \sigma_3(L) \geq \frac{1}{3})$, and $(\frac{3}{7}A + \frac{4}{7}D, \frac{1}{3}a + \frac{2}{3}d, \frac{1}{3}L + \frac{2}{3}R)$.

Problem 6 Compute all Nash equilibria of the symmetric normal form game in figure 4:

			2	
		L	C	R
	T	$0, 0$	$6, -3$	$-4, -1$
1	M	$-3, 6$	$0, 0$	$5, 3$
	B	$-1, -4$	$3, 5$	$0, 0$

Figure 4: game for problem 6

See figures for pictures of the best response correspondences. To compute Nash equilibria, we'll look at each possible supports for player 1:

T: 2 prefers L. Given this, 1 prefers T. Equilibrium.

M: 2 prefers L. Given this, 1 prefers T. No equilibrium.

B: 2 prefers C. Given this, 1 prefers T. No equilibrium.

TM: 2 prefers L. Given this, 1 prefers T only. No equilibrium.

TB: 2 could play L, R, C, LR or RC, depending on the mixture. Look at each of these severally:

- L: 1 prefers T only. No equilibrium.
- R: 1 prefers M. No equilibrium.
- C: 1 prefers T only. No equilibrium.
- LR: requires 1 playing $\frac{4}{5}T + \frac{1}{5}B$. 1 is willing to do this if 2 plays $\frac{4}{5}L + \frac{1}{5}R$. Equilibrium.
- RC: Given this support, 1 prefers either T, M, or a mixture of T and M. No equilibrium.

MB: For 1 to be willing to mix between M and B, 2 must assign prositive probability to R, but 2 will not do this if 1 mixes between M and B.

TMB: equilibrium with both putting $\frac{1}{3}$ probability on each of their strategies.

So there are 3 Nash equilibria: (T, L), $(\frac{4}{5}R + \frac{1}{5}B, \frac{4}{5}L + \frac{1}{5}R)$, and $(\frac{1}{3}T + \frac{1}{3}M + \frac{1}{3}B, \frac{1}{3}L + \frac{1}{3}C + \frac{1}{3}R)$.

Problem 7 Consider the following payoffs for player 1 (player 2's payoffs are irrelevant to this question):

We saw in class that all mixtures of T and M are strictly dominated by some other strategy. Identify, for each mixture of T and M, a strategy that strictly dominates that mixture. That is, for the mixture $\alpha T + (1 - \alpha)M$, there is some strategy $\sigma(\alpha) \in \Sigma_1$ which strictly dominates $\alpha T + (1 - \alpha)M$, and which does not put positive probability on both T and M. Find $\sigma(\alpha)$ for all $\alpha \in (0, 1)$.

		2	
		l	r
	T	$3, \cdot$	$0, \cdot$
1	M	$0, \cdot$	$3, \cdot$
	B	$2, \cdot$	$2, \cdot$

Figure 5: game for problem 7

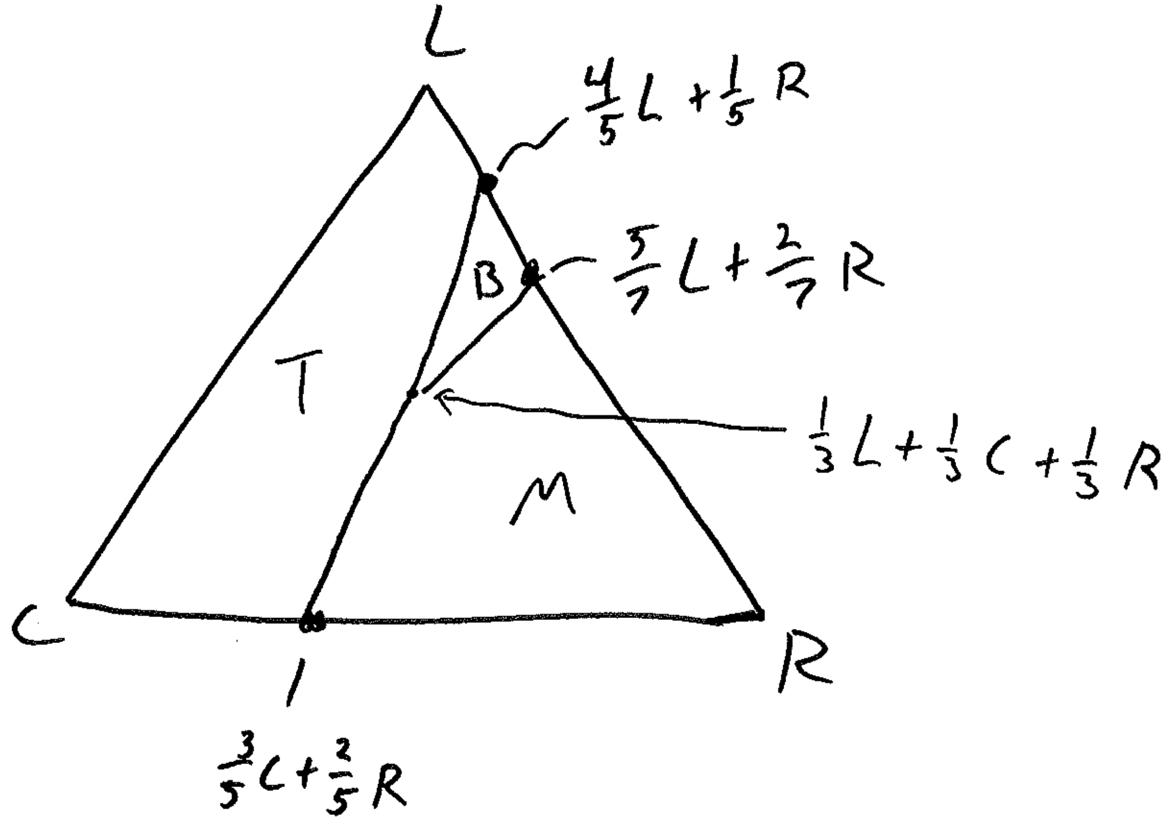
The strategy $\sigma_\alpha = \alpha T + (1-\alpha)M$ is strictly dominated by the strategy B if $\alpha \in (\frac{1}{2}, \frac{2}{3})$. To see this, note that $u_1(\sigma_\alpha, \sigma_2) \in (1, 2)$ for all $\alpha \in (\frac{1}{2}, \frac{2}{3})$ and all $\sigma_2 \in \Sigma_2$.

Now suppose that $\alpha < \frac{1}{3}$. Consider the strategy $\sigma_p = p(\alpha)B + (1 - p(\alpha))M$, and set $p(\alpha) = 2\alpha$. If player 2 plays l, 1 gets a payoff of 4α from σ_p versus 3α from σ_α. If player 2 plays r, player 1 gets $3 - 2\alpha$ from σ_p versus $3 - 3\alpha$ from σ_α. Figure 7 illustrates this; since 1 gets a higher payoff from σ_p than σ_α both when 2 plays l and when 2 plays r, σ_p must also give the higher payoff for all mixed strategies of player 2.

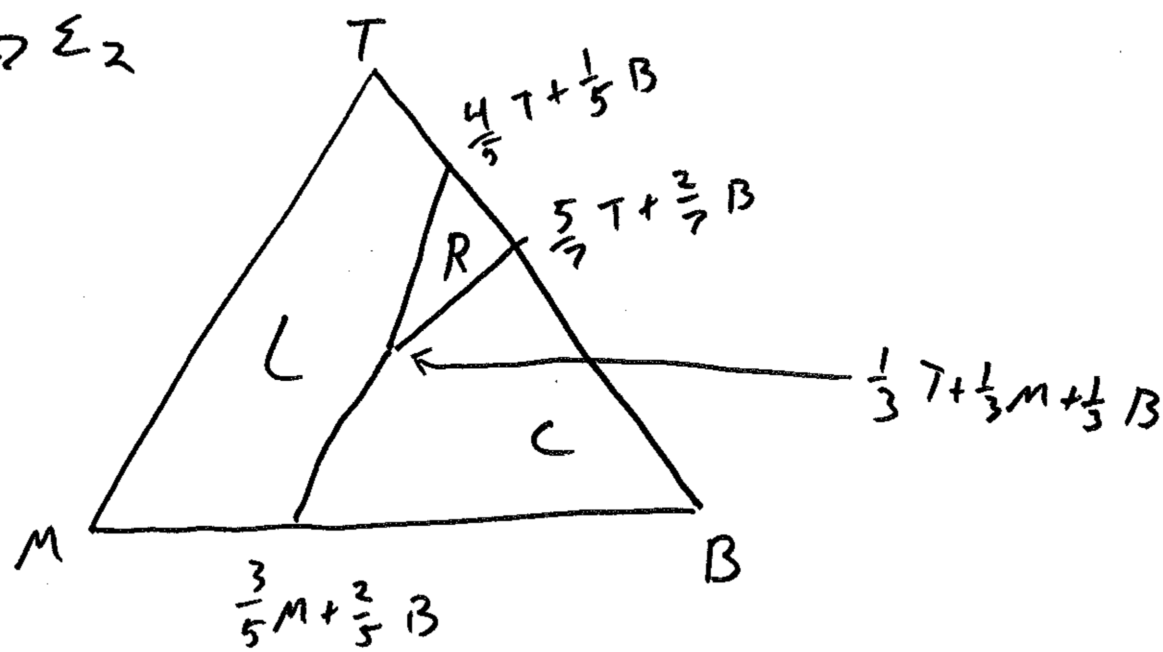
If $\alpha \geq \frac{1}{3}$, let $\sigma_q = q(\alpha)B + (1 - q(\alpha))T$, and set $q(\alpha) = 2\alpha$, and the same result holds.

Therefore, σ_α is strictly dominated by B for $\alpha \in (\frac{1}{3}, \frac{2}{3})$, by σ_p for $\alpha \in [0, \frac{1}{3}]$ and by σ_q for $\alpha \in [\frac{1}{3}, 1]$.

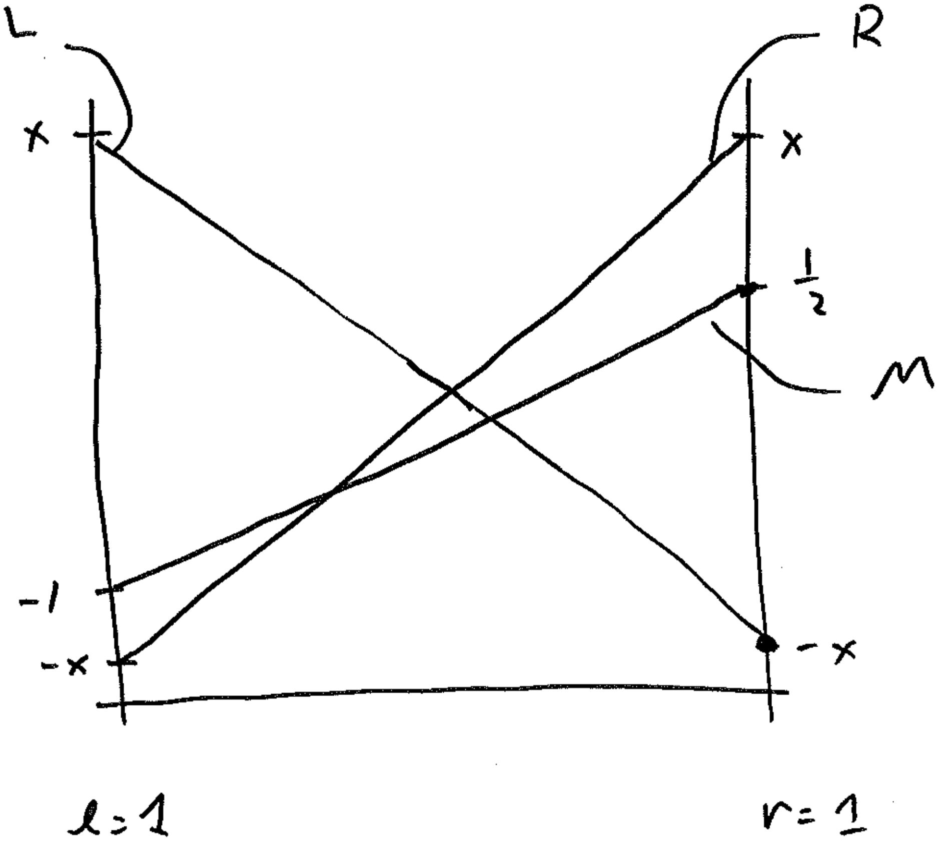
$BR_1: \Sigma_2 \Rightarrow \Sigma_1$



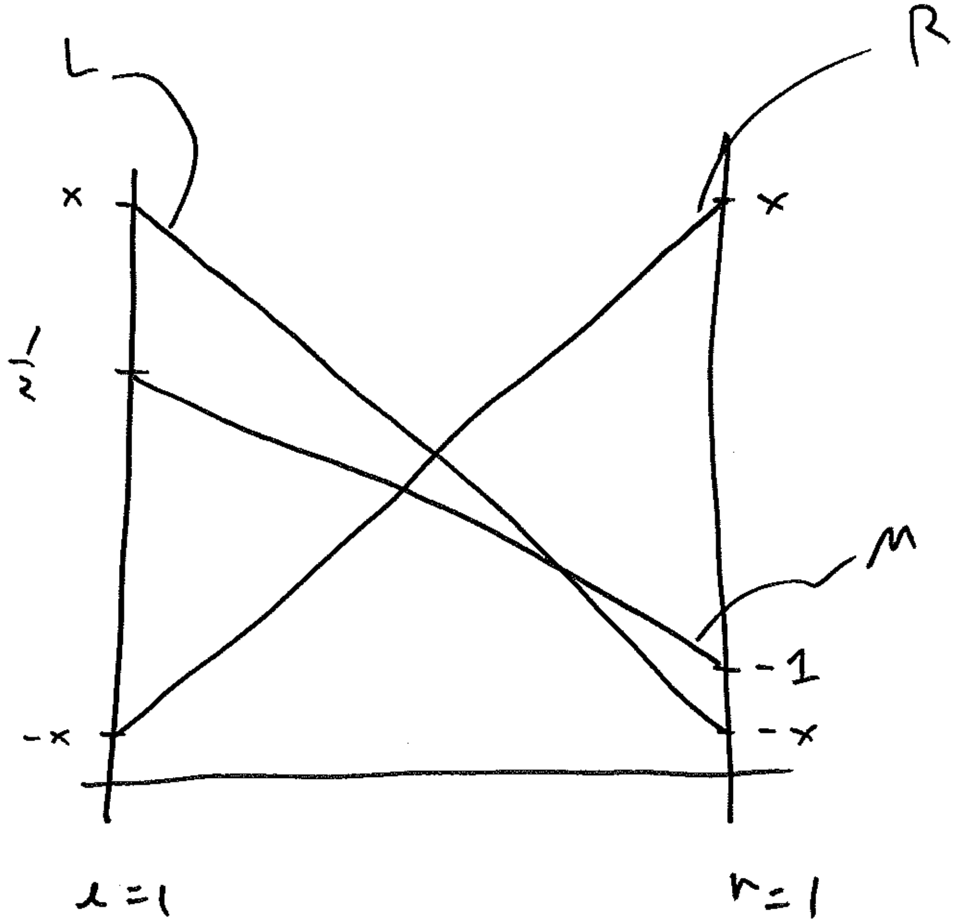
$BR_2: \Sigma_1 \Rightarrow \Sigma_2$



2 plays U



2 plays D



1's utility as function of 2,3's strategy.

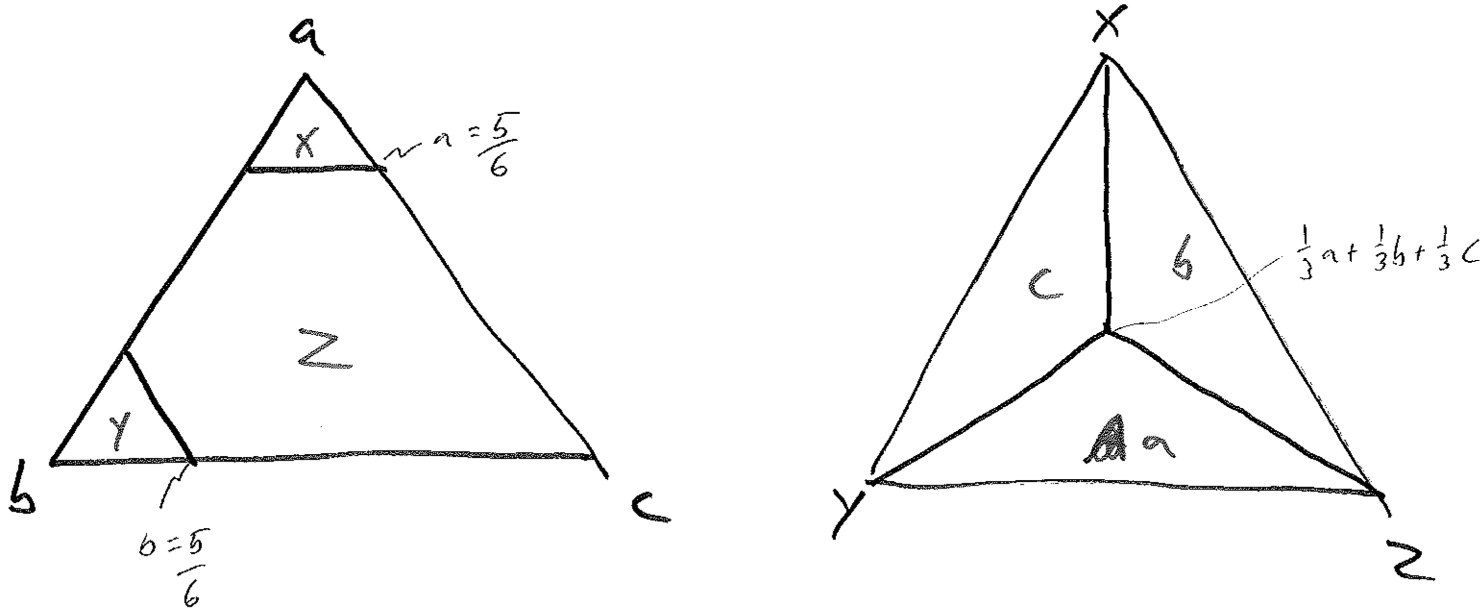


figure 3

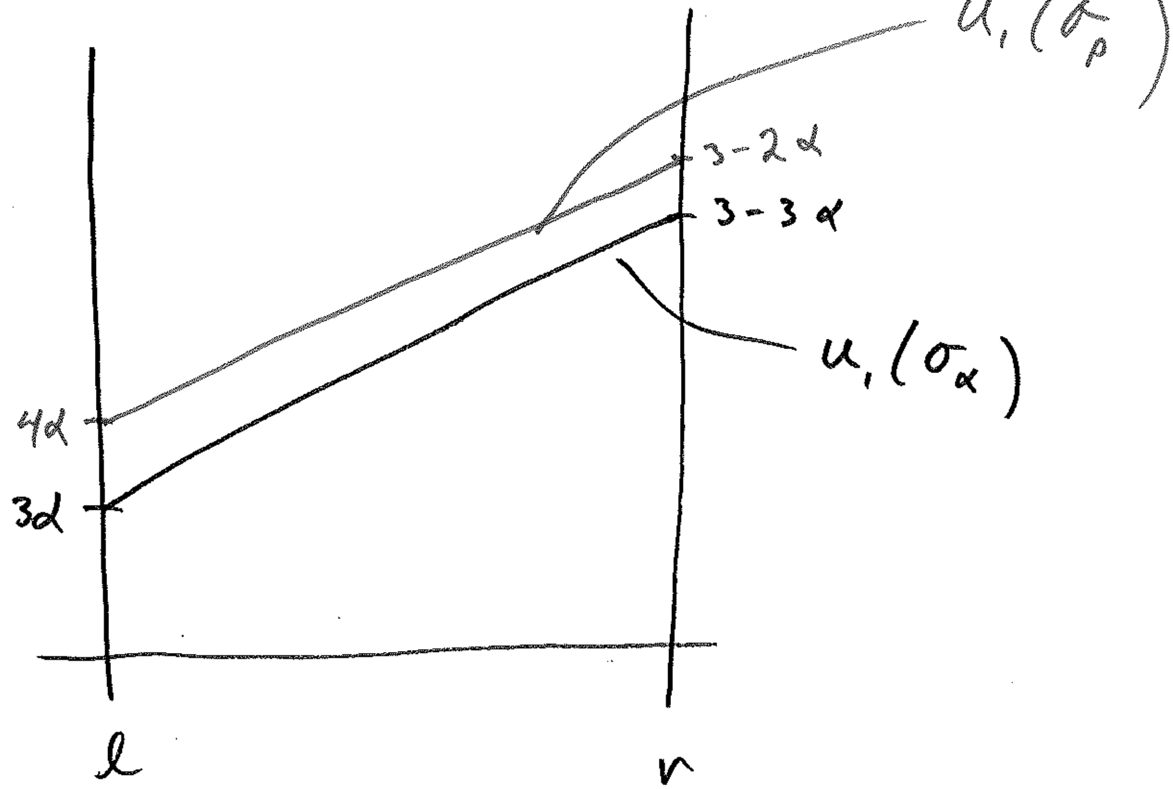


figure 7