Homework 4

answers

Problem 1 Consider infinite repetitions of the games in figures 1, 2, and 3. In each case, sketch the set of payoff vectors attainable in a subgame perfect equilibrium, and those which are attainable in a subgame perfect equilibrium using Nash reversion strategies. (Suppose that the discount rate is very close to one.)

Figure 1: See sketch below. All feasible, individually rational payoffs are supportable in a subgame perfect equilibrium if δ is high enough, while the stage game's sole Nash equilibrium has payoffs $(2, 2)$. As there are no feasible payoff vectors that dominate $(2, 2)$, Nash reversion is useless in this game. Note that 1 and 2 both have minmax payoffs of 1.

Figure 2: 1's minmax payoff is 1, 2's is 0. The stage game's lone Nash equilibrium is $(\frac{1}{2}T + \frac{1}{2}B, \frac{1}{2}l + \frac{1}{2}r)$, which gives payoffs $(2, 2)$. See sketch below for a picture of payoffs supportable using Nash reversion and supportable in some SPE using carrot and stick strategies, if δ is high enough..

Figure 3: 1's minmax payoff is $\frac{7}{2}$, while 2's is $\frac{1}{2}$. There is one Nash equilibrium, at (T, R), with payoffs $(4, 1)$. See sketch below for strategies supportable in a SPE using Nash reversion and carrot and stick strategies, if δ is high enough..

1 \ 2	A	B
a	$2, 2$	$1, 0$
b	$0, 1$	$0, 0$

Figure 1: Normal form game 1

1 \ 2	L	R
T	$3, 0$	$1, -2$
B	$5, 4$	$-1, 6$

Figure 2: Normal form game 2

1 \ 2	L	R
T	$2, -2$	$4, 1$
M	$1, 3$	$0, 0$
B	$5, 3$	$3, 4$

Figure 3: Normal form game 3

Problem 2 Consider the normal form game G in figure 4 below.

a. Determine the set of Nash equilibria of the normal form game (note that the game is zero-sum).

Since G is a zero-sum game, the easiest way to find its Nash equilibria is to look for the strategies 2 uses to minmax 1. From inspection of player 1's best response correspondence, this is evidently at $\frac{5}{11}X + \frac{5}{11}Y + \frac{1}{11}Z$, giving 1 a minmax payoff of $\frac{51}{11}$. For 2 to be willing to mix between X, Y, and Z, 1 evidently needs to play the mixture $\frac{13}{77}B + +\frac{8}{77}C + \frac{8}{11}E$. Conclude there is a unique Nash equilibrium, at $(\sigma_1, \sigma_2) = (\frac{13}{77}B + +\frac{8}{77}C + \frac{8}{11}E, \frac{5}{11}X + \frac{5}{11}Y + \frac{1}{11}Z)$.

b. Let $G^\infty(\frac{3}{4})$ be an infinite repetition of G with common discount rate $\frac{3}{4}$. Sketch both the set of feasible payoffs of $G^\infty(\frac{3}{4})$ and the set of payoffs which are sustainable in some subgame perfect equilibrium of $G^\infty(\frac{3}{4})$.

In this special case of a zero-sum game, the set of feasible payoffs is a straight line, and so the set of feasible, individually rational payoffs is a singleton. Therefore, the set of payoffs attainable in an equilibrium of the repeated game is no different from the set of payoffs attainable in the one-shot game.

Problem 3 Consider an infinite repetition of the normal form game in figure 5. For what values of δ can the play path $\{(C, C), (C, C), ...\}$ be supported in a subgame perfect equilibrium?

Since the game is symmetric, we need check incentive compatibility for only one player. Under grim trigger strategies with mutual minmaxing as the punishment path, player 1 prefers not to deviate if

$$2 \geq (1 - \delta)8 + \delta 1$$
$$\iff \delta \geq \frac{6}{7}$$

Since mutual minmaxing coincides with a Nash equilibrium (a unique feature of the prisoner's dilemma), in this case no player will want to deviate from a punishment path of perpetual play of the Nash equilibrium (D, D), and so the grim trigger strategy is also a SPE so long as $\delta \geq \frac{6}{7}$.

1 \ 2	X	Y	Z
A	10, 0	0, 10	0, 10
B	9, 1	1, 9	1, 9
C	2, 8	8, 2	1, 9
D	2, 8	2, 8	7, 3
E	4, 6	5, 5	6, 4

Figure 4: Normal form game 4

1 \ 2	C	D
C	2, 2	0, 8
D	8, 0	1, 1

Figure 5: Normal form game 5

Problem 4 You and a friend are computing the subgame perfect equilibria of an infinitely repeated game. Your friend suggests that to perform the computation efficiently, you should begin by eliminating all dominated actions from the stage game. Evaluate your friend's suggestion.

This is a bad idea; consider the prisoner's dilemma (for example, the one in figure 5 of this homework). C is a strictly dominated strategy for both players, yet eliminating it for both players will get rid of all but one payoff pair (1,1), including every point on the Pareto frontier.

Problem 5 MWG problem 12.AA.1

Suppose we had a SPNE where player 1's repeated game payoff was $\underline{\pi}_1 < \pi_1^*$, with Nash reversion as the punishment path. Player 1 will not deviate from the equilibrium path if and only if the following inequality holds, where $\overline{\pi}_1$ is his highest possible stage game deviation from the equilibrium path:

$$\underline{\pi}_1 \geq (1-\delta)\overline{\pi}_1 + \delta\pi_1^* \tag{1}$$

Clearly, (1) does not hold, as both $\overline{\pi}_1$ and π_1^* are larger than $\underline{\pi}_1$.

Problem 6 MWG problem 12.B.1

a. The monopolist's FOC is $\frac{\partial p}{\partial q}q + p - c'(q) = 0$. In other words, $p - c'(q) = -q\frac{\partial p}{\partial q} = -\frac{q}{\frac{\partial q}{\partial p}}$. Therefore,

$$\frac{p - c'(q)}{p} = -\frac{1}{\frac{\partial 1}{\partial p}\frac{p}{q}} = -\frac{1}{\epsilon^D} \tag{2}$$

b. It suffices to show that marginal revenue, $\frac{\partial p}{\partial q}q + p$, is negative iff demand is elastic, as in this case the monopolist unambiguously increases profits by decreasing quantity (as this causes revenue to increase and costs to decrease). Now, supposing $\frac{\partial q}{\partial p}\frac{p}{q} < -1$,

$$\begin{aligned} MR = \frac{\partial p}{\partial q}q + p < 0 \\ \iff p < -q\frac{1}{\frac{\partial q}{\partial p}} \\ \iff \frac{\partial q}{\partial p}\frac{p}{q} < -1 \end{aligned} \tag{3}$$

Therefore, if demand is elastic, the monopolist strictly increases profits by decreasing quantity, so the profit-maximizing quantity can not possibly be at a location where demand is elastic.

Problem 7 MWG problem 12.B.8

a. The monopolist's profit-maximization problem is:

$$\max_{q_q, q_2} (a - bq_1)q_1 + (a - bq_2)q_2 - c_1q_1 - (c_1 - mq_1)q_2 \tag{4}$$

which has first order conditions:

$$\begin{aligned} q_1 &: \quad a - 2bq_1 - c_1 + mq_2 = 0 \\ q_2 &: \quad a - 2bq_2 - c_1 = 0 \end{aligned}$$

Solving the above system, we get:

$$q_2 = \frac{a - c_1}{2b}$$
$$q_1 = \frac{a - c_1}{2b}\left(1 + \frac{m}{2b}\right)$$

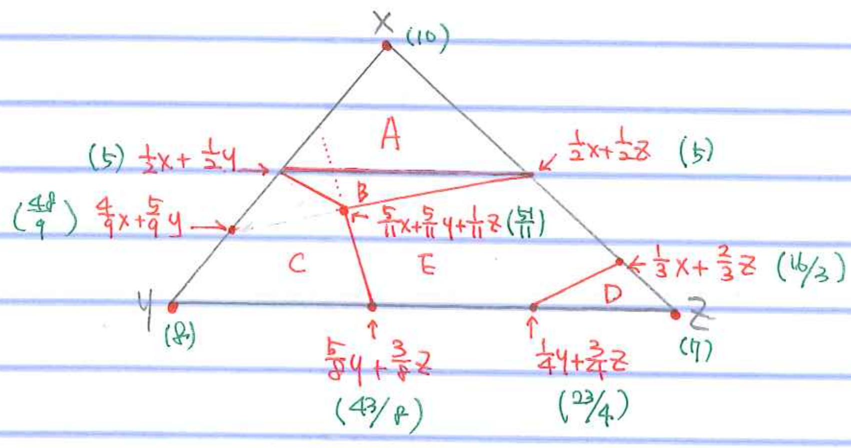
b,c. Clearly, $q_2 = \frac{a-c_1+mq_1}{b}$ (this is where the demand curve crosses the marginal cost curve). Interestingly, $q_1 > \frac{a-c_1}{b}$, as the planner has an incentive to overproduce in period 1 to lower period 2 costs. Period 2 surplus is $\frac{1}{2}(a - c_1)\left(\frac{a-c_1+mq_1}{b}\right)$. Period 1 surplus is equal to a constant minus the deadweight loss of overproduction, or $\frac{1}{2}\left(q_1 - \frac{a-c}{b}\right)(c_1 - (a - bq_1))$. The planner then solves:

$$\max_{q_1} \frac{1}{2}(a - c_1)\left(\frac{a - c_1 + mq_1}{b}\right) - \frac{1}{2}\left(q_1 - \frac{a - c}{b}\right)(c_1 - (a - bq_1)) \tag{5}$$

which has FOC:

$$q_1 = \frac{a - c_1}{b}\left(1 + \frac{m}{2b}\right)$$

This amounts to a $MB = MC$ calculation, but the marginal benefit includes greater surplus next period.


x (10)
A
(5) 1/2x + 1/2y
1/2x + 1/2z (5)
B
(48/9) 4/9x + 5/9y
5/11x + 5/11y + 1/11z (51/11)
C
E
1/3x + 2/3z (16/3)
D
y (8)
z (7)
5/8y + 3/8z (43/8)
1/4y + 3/4z (23/4)